

THE PHOTODISINTEGRATION OF THE LIGHTEST

NUCLEI

I should like to thank Professor J. A. ...
suggesting this problem and for all ...
also much indebted to **A. C. Douglas** ...
discussions and for considerable assistance with ...
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like to express my gratitude to the ...
and Industrial Research for the award of a ...

Thesis presented to the University of Glasgow
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CHAPTER 1.

Introductory Remarks.

1. The Photodisintegration of the Lightest Nuclei.

The basic theory of photonuclear reactions is an application of the quantum mechanical theory of the interaction of charged particles with the electromagnetic field. The validity of this theory has been well demonstrated in calculations on the emission and absorption of photons by electrons bound in atoms.

The application of this theory to photonuclear reactions has been made with considerable success in the photodisintegration of the deuteron. However, since the detailed study of the deuteron and of the two-nucleon scattering data has not yet yielded a satisfactory nuclear interaction, it is of interest to investigate further nuclear problems involving more than two nucleons, so that the adequacy or inadequacy of the proposed two-body interaction for such problems may be demonstrated.

The theory tests not only the nuclear potential but offers a test of Siegert's theorem, i.e. a test of whether the theory successful in the atomic case, still applies when meson exchange is taking place. It is known that the theory breaks down at high energies where the effect takes place by way of the virtual production of photopions. The study of photonuclear reactions should give the range of validity of Siegert's theorem.

As will be seen later, the theory gives a method of exploring nuclear wave functions without an explicit knowledge of the nuclear force, provided that the plane wave approximation for the final state of the reaction, is valid.

Apart from the study of the deuteron, the photodisintegration of the three particle nuclei ${}^3\text{H}$ and ${}^3\text{He}$ is simplest to study theoretically. However, the experimental data on these reactions is very meagre and so the following investigation will chiefly be concerned with the study of the photodisintegration of the α -particle and of the inverse reaction, radiative capture.

The study of this reaction is of considerable importance in the development of the theory of photonuclear reactions for the following reason.

The investigations of high-energy photoprotons and photoneutrons emitted from nuclei have demonstrated many characteristics of photonuclear reactions. Several different models of photon-nucleon interaction have been proposed to explain these different phenomena. In particular, for photon energies $E_\gamma > 100$ Mev, the experiments are satisfactorily explained by the quasi-deuteron model (Levinger 1951). At such high γ -ray energies the wavelength is less than the inter-nucleon distance. The γ -ray will therefore prefer to interact with only one nucleon at a time. Since a high energy photon has relatively little momentum, its energy must be shared by at least two nucleons, if energy and momentum are to be conserved. Hence photodisintegration can only take place when two nucleons occur within a wavelength. The

electric dipole term is predominant in the photoeffect at high photon energies (as will be seen later). Since the dipole moment of a pair of protons is zero, the main contribution to the photoeffect is found when a neutron and a proton occur in close proximity. This gives rise to the name of the "quasi-deuteron" model.

However at lower energies Yoshida (1951) has shown that the angular distributions of photoprotons are in better agreement with absorption in nuclear sub-units of four nucleons; i.e. an α -particle model. This model was first suggested by Levinger and Bethe (1950) to account for the fact that the mean square displacement $\langle r^2 \rangle_{00}$ of a nucleon in the ground state of a nucleus, does not vary greatly with the mass number A. This is borne out by the calculations of Goldenberg and Lopes (1954). They find that the value of $\langle r^2 \rangle_{00}$ in the range $A = 12$ to $A = 209$ is practically constant and is of the order of that for the α -particle. In order to develop this model further, it is important to investigate the photodisintegration of free α -particles.

2. Multipole Radiation Theory.

The initial step in the theory involves the separation of the electromagnetic field into the various electric and magnetic multipole fields. This has been considered by several authors; e.g. Blatt and Weisskopf (1952), Sachs and Austern (1951) and Moskowski (1955). Here the method of MacDonald (1955) will be followed. As there are several misprints in that paper, some of the details will be repeated here.

It is usual to start from the Schroedinger equation for a charged particle in an electromagnetic field. The non-relativistic Schroedinger

Hamiltonian for the interaction of a system of A nucleons with the electromagnetic field can be written in the isotopic spin formalism, correct to first order in the vector potential \underline{A} as

$$H_I = - \sum_{i=1}^A \left\{ \frac{e}{mc} \underline{p}_i \cdot \underline{A}(\underline{r}_i) \left(\frac{1}{2} - t_{g_i} \right) + \left[\mu_p \left(\frac{1}{2} - t_{g_i} \right) + \mu_N \left(\frac{1}{2} + t_{g_i} \right) \right] \underline{g}_i \cdot [\underline{\nabla} \times \underline{A}(\underline{r}_i)] \right\} \quad \dots\dots\dots(1)$$

where the isotopic spin operator t_g has the eigenvalues $(+\frac{1}{2})$ for a neutron and $(-\frac{1}{2})$ for a proton. In (1), m is the nucleon mass, \underline{r}_i the position vector of the i^{th} nucleon, \underline{p}_i its momentum and \underline{g}_i the Pauli spin vector. μ_p and μ_N are respectively, the proton and neutron magnetic moments.

This form of Hamiltonian is dependant explicitly on the nucleon variables only. However, the exchange forces between neutron and proton imply the existence of a current of charged pions between the nucleons. Exact calculations should include the effects of these pion currents.

However, it has been shown by Siegert (1937) and more generally by Sachs and Austern (1951) that the electric moment operators are independent of the form of the Hamiltonian. This result implies that the electric multipole moment operators are independent of the pion exchange currents and is known as Siegert's theorem.

Hence predictions concerning the electric moments and electric transition probabilities can be made safely even in the absence of a full knowledge of the exchange effects.

It should be noticed that Siegert's theorem is only strictly true in the "long-wavelength" approximation (Brennan and Sachs 1952).

The possibility of pion photoproduction should have a marked influence on the nuclear photodisintegration at photon energies of the order of 140 Mev or greater. The separation of this influence from the "ordinary" process of photodisintegration can be accomplished only if a reliable theoretical value for the cross-section of the "ordinary" process is available. Any deviation from the calculated curve can be interpreted as an indication of effects which depend explicitly on the pion variables.

The photoelectric cross-section for the ordinary process is normally derived by assuming Siegert's theorem. Brennan and Sachs have shown that the theorem is not valid at high energies; in fact, for the deuteron it breaks down in the neighbourhood of 50 Mev. However these authors have shown that the errors are still quite small at 300 Mev.

Thus Siegert's theorem is only strictly true for energies less than 50 Mev. At higher energies, the meson effects must be included explicitly by the introduction of additional terms in the Hamiltonian(1).

For simplicity a single charged particle will be considered. Then

$$H_I = - \frac{e}{mc} \mathbf{p} \cdot \mathbf{A} - \mu \mathbf{S} \cdot (\nabla \times \mathbf{A}) \dots\dots(2)$$

where \mathbf{A} satisfies the gauge condition $\nabla \cdot \mathbf{A} = 0$.

The transition probability per unit time for the emission of radiation is given by perturbation theory (Schiff 1949) as

$$T_{a \rightarrow b} = \frac{2\pi}{\hbar} / H_{ab}^2 \rho(E). \quad \dots\dots(3)$$

where a,b are indices denoting the initial and final states ϕ_a and ϕ_b respectively and $\rho(E)$ is the density of final states per unit energy. The field may be restricted to one varying periodically with time without loss of generality

$$\underline{A}(\underline{r}, t) = \underline{A}(\underline{r}) e^{-i\omega t} + \underline{A}^*(\underline{r}) e^{i\omega t} \quad \dots\dots(4)$$

Then

$$H_{ab} = \int \phi_a^* \underline{A}(\underline{r}) \phi_b d\tau \quad \dots\dots(5)$$

The vector potential is then expanded into a series by the use of the vector spherical harmonics $Y_{JL1}^M(\theta, \phi)$ as defined by Blatt and Weisskopf (1952). The Y_{JL1}^M are eigenfunctions of the angular momentum operators J^2 and J_z where

$$\underline{J} = \underline{L} + \underline{S} \quad \underline{L} = -i \hbar \underline{r} \times \underline{\nabla} \quad \underline{S} = i \hbar \underline{e}_k \times$$

and \underline{e}_k is the unit vector along the x_k -axis.

Since the vector spherical harmonics form a complete set, the vector field $\underline{A}(\underline{r})$ can be expanded as

$$\underline{A}(\underline{r}) = \sum_{J=0}^{\infty} \sum_{M=-J}^{+J} \underline{A}(J, M; \underline{r}) \quad \dots\dots(6)$$

where

$$\underline{A}(J, M; \underline{r}) = \frac{1}{r} \left\{ f_{JM}(r) \underline{X}_{JM} + \underline{\nabla} \times [g_{JM}(r) \underline{X}_{JM} + h_{JM}(r) \underline{X}_{JM}] \right\}$$

..... (7)

with

$$\underline{X}_{JM}(\theta, \phi) \equiv Y_{JJ1}^M(\theta, \phi) \dots\dots\dots(8)$$

Hence the ~~Each~~ term $\underline{A}(J, M; \underline{r})$ in the expansion (6) is called a pure multipole field.

In future the quantum numbers l, m will be used instead of J, M in the multipole expansion.

The electromagnetic multipole fields are further subdivided into electric and magnetic multipoles. This division is made on account of the parity of the field. It is convenient to define the parity of a multipole field as the parity of the magnetic field $\underline{H}(\underline{r})$.

In electric multipole radiation of order (l, m) , the electric field has parity $-(-1)^l$, the magnetic field parity $(-1)^l$. In magnetic multipole radiation of order (l, m) the parities are exactly the opposite.

Hence

$$\text{Parity of electric multipole } (l, m) = (-1)^l \dots\dots\dots(9)$$

$$\text{Parity of magnetic multipole } (l, m) = -(-1)^l$$

Now, Maxwell's equations give:-

$$\underline{H}(\underline{r}) = \text{curl } \underline{A}(\underline{r}) \quad ; \quad \underline{E}(\underline{r}) = i k \underline{A}(\underline{r})$$

where the time dependence has again been removed and where the wave number k is given by

$$k = \frac{\omega}{c} \dots\dots\dots(10)$$

The first term in (7) has parity $(-1)^l$. Hence it is associated with a magnetic field of parity $-(-1)^l$. The term in \underline{X}_{lm} is therefore

a magnetic multipole field A_M . The remaining terms in (7) have parity $-(-1)^l$ and are associated with a magnetic field of parity $(-1)^l$.

Hence the terms in $\nabla \times \underline{X}_{lm}$ provide an electric multipole field A_E .

If $\underline{A}_E(\underline{r})$ and $\underline{A}_M(\underline{r})$ are now normalised to energy $(\hbar c k)$ inside a sphere of radius a , on the surface of which \underline{A} vanishes, then the vector potentials of the multipole fields are given by

$$\underline{A}_E(l,m) = \frac{1}{k} \left(\frac{\hbar c k}{a} \right)^{\frac{1}{2}} \frac{[\nabla \times \underline{L}]^{\frac{1}{2}}}{[l(l+1)\hbar^2]^{\frac{1}{2}}} j_l(kr) Y_{lm}(\theta\phi) \dots (11)$$

$$\underline{A}_M(l,m) = i \left(\frac{\hbar c k}{a} \right)^{\frac{1}{2}} \frac{\underline{L}}{[l(l+1)\hbar^2]^{\frac{1}{2}}} j_l(kr) Y_{lm}(\theta\phi) \dots (12)$$

where $j_l(kr)$ is the spherical Bessel function and $Y_{lm}(\theta\phi)$ the usual scalar spherical harmonic.

The matrix element H_{ab} can now be written as

$$H_{ab} = -\frac{1}{c} \int \underline{J}_{ab} \cdot \underline{A} d\tau - \mu \int \phi_a^x \underline{6} \cdot (\nabla \times \underline{A}) \phi_b d\tau$$

where

$$\underline{J}_{ab} = \left[\frac{e}{2mc} \right] \times \left[\phi_a^x (\underline{p} \phi_b) + (\underline{p} \phi_a)^x \phi_b \right]$$

is the quantum mechanical transition current.

The generalised multipole moments are now defined by

$$\begin{aligned} \text{Electric: } -\underline{Q}_{lm} &= \frac{(2l+1)!!}{(l+1)k^l} \left\{ e \int \left[\frac{d}{dr} r j_l(kr) Y_{lm}(\theta\phi) \right] \phi_a^x \phi_b d\tau \right. \\ &\quad \left. + \frac{ik}{c} \int j_l(kr) Y_{lm}(\theta\phi) \underline{r} \cdot \underline{J}_{ab} d\tau \right\} \dots (13) \end{aligned}$$

$$2_{lm}^1 = \frac{(2l+1)!!}{(l+1)k!} \left\{ -i \mu k \int j_l(kr) Y_{lm}(\theta) \operatorname{div} (\phi_a^x \underline{r} \times \underline{\phi}_b) d\tau \right\}$$

Magnetic:-

$$m_{lm} = - \frac{(2l+1)!!}{(l+1)k!} 2 \mu \int j_l(kr) Y_{lm}(\theta) \operatorname{div} (\phi_a^x \underline{L} \phi_b) d\tau \dots\dots\dots (14)$$

$$m_{lm}^1 = - \frac{(2l+1)!!}{(l+1)k!} \mu \left\{ \int \left[\frac{d}{dr} r j_l(kr) Y_{lm}(\theta) \operatorname{div} (\phi_a^x \underline{\phi}_b) d\tau - k^2 \int j_l(kr) Y_{lm}(\theta) (\phi_a^x \underline{\phi} \cdot \underline{r} \phi_b) d\tau \right] \right\}$$

The transition probabilities per unit time (integrated over solid angle) are

$$T_{ab}^E = \frac{8\pi(l+1)}{l!(2l+1)!!}]^2 \frac{k^{2l+1}}{\hbar} / 2_{lm} + 2_{lm}^1 /^2 \dots\dots\dots (15)$$

$$T_{ab}^M = \frac{8\pi(l+1)}{l!(2l+1)!!}]^2 \frac{k^{2l+1}}{\hbar} / m_{lm} + m_{lm}^1 /^2 \dots\dots\dots (16)$$

These formulae are exact in that no assumption has been made about the magnitude of KR where R is the nuclear dimension.

Since for small kr

$$j_l(kr) = \frac{(kr)^l}{(2l+1)!!} - \frac{(kr)^{l+2}}{(2l+3)!!} 2 + \dots\dots\dots (17)$$

2_{lm} (for example) may be expanded as

$$2_{lm} = Q_{lm}^{(0)} + Q_{lm}^{(1)} + \dots\dots\dots (18)$$

where the terms correspond to the successive terms in (17).

In the "long-wavelength" approximation $KR \ll 1$,

$$Q_{lm}^{(0)} = e \int r^l Y_{lm}(\theta\phi) \phi_a^x \phi_b \, d\tau \quad \dots\dots\dots(19)$$

$$Q_{lm}^{(0)1} = -\frac{ik}{l+1} \mu \int r^l Y_{lm}(\theta\phi) \operatorname{div}(\phi_a^x \underline{r} \times \underline{\phi}_b) \, d\tau$$

$$M_{lm}^{(0)1} = -\mu \int r^l Y_{lm}(\theta\phi) \operatorname{div}(\phi_a^x \underline{\phi}_b) \, d\tau$$

$$M_{lm}^{(0)} = -\frac{2\mu}{l+1} \int r^l Y_{lm}(\theta\phi) \operatorname{div}(\phi_a^x \underline{L} \phi_b) \, d\tau \quad \dots\dots\dots(20)$$

These are precisely the single particle moments of Blatt and Weisskopf (1952).

From the above, it is seen that one quantum of multipole radiation (1,m) carries with it, an angular momentum 1, with Z component m. (both measured in units of \hbar). If the quantum is emitted by a nucleus going from a state ϕ_a to a state ϕ_b , then the following selection rules are obeyed.

Angular Momentum

$$|J_a - J_b| \leq 1 \leq J_a + J_b \quad J_a = 0 \rightarrow J_b = 0 \text{ forbidden}$$

$$M_a - M_b = m \quad \dots\dots\dots(21)$$

Parity

	Electric	Magnetic
1 even	$\Pi_a = \Pi_b$	$\Pi_a = -\Pi_b$
1 odd	$\Pi_a = -\Pi_b$	$\Pi_a = \Pi_b \quad \dots\dots\dots(22)$

It can be shown from general considerations (e.g. Blatt and Weisskopf 1952) that the emission probability of multipole quanta 1, is a rapidly

decreasing function of l . Also for the same multipole order l , the intensity of magnetic radiation is smaller than the intensity of electric radiation by a factor of order $\frac{v^2}{c^2}$. For a nucleus this is of order of 0.01.

In the following, the standard notation $E l (Ml)$ for electric (magnetic) 2^l -pole radiation will be used.

MacDonald (1955) has estimated the effect of two types of higher order terms which will contribute to $E1$ radiation. He finds:-

$$\left/ \frac{Q_{1m}^{(0)}}{Q_{1m}^{(0)}} \right/ ^2 \sim \frac{1}{25} \left(\frac{h\omega}{Mc^2} \right)^2 \sim 4 \cdot 10^{-4} \text{ for 100 Mev } \gamma\text{-rays.}$$

and

$$\left/ \frac{Q_{1m}^{(1)}}{Q_{1m}^{(0)}} \right/ ^2 \sim \left(\frac{2}{15} \right)^2 (kR)^4 \sim 10^{-2} \text{ for 100 Mev } \gamma\text{-rays.}$$

Hence it is seen that the contributions of both $Q_{1m}^{(0)}$ and $Q_{1m}^{(1)}$ are small, even at 100 Mev and may therefore be neglected.

It is sufficient to use the "long-wavelength" approximation.

The Hamiltonian (1) can be separated as follows:-

$$H_I = H_O + H_1 \quad \dots\dots\dots(23)$$

where

$$H_O = - \sum_{i=1}^A \left\{ \frac{e}{2mc} \underline{p}_i \cdot \underline{A}(\underline{r}_i) + \frac{1}{2} (\mu_p + \mu_N) \underline{S}_i \cdot [\underline{V} \times \underline{A}(\underline{r}_i)] \right\} \quad \dots\dots\dots(24)$$

$$H_1 = \sum_{i=1}^A \left\{ \frac{e}{2mc} \underline{p}_i \cdot \underline{A}(\underline{r}_i) + (\mu_p - \mu_N) \underline{S}_i \cdot [\underline{V} \times \underline{A}(\underline{r}_i)] \right\} t_{g_i} \quad \dots\dots\dots(25)$$

H_0 is a scalar in isotopic spin space whilst H_1 is the g component of a vector. For an E1 transition due to H_0 , the first order term in the transition operator A is derived from the term $\left\{ \frac{e}{2mc} \mathbf{p}_i \cdot \mathbf{A}(\mathbf{r}_i) \right\}$ in (24), and is given by $\sum_{i=1}^A \mathbf{r}_i \cdot \mathbf{A}(\mathbf{r}_i)$. To allow for the nuclear recoil, the centre of mass of the nucleus is taken as the origin and therefore $\sum_{i=1}^A \mathbf{r}_i = 0$. Hence the probability of E1 transitions induced by H_0 is zero. Then only H_1 can produce transitions.

For an E1 transition due to H_1 , the first order term is given by $\sum_{i=1}^A \mathbf{r}_i \cdot \mathbf{t}_{g_i} \neq 0$. Since H_1 is the g -component of a vector in isotopic spin space, an E1 transition must cause a change of one unit (vectorially) in the isotopic spin. This leads to the selection rules

$$\begin{aligned} |T_a - T_b| &\leq 1 \leq T_a + T_b \\ T_a = 0 &\longrightarrow T_b = 0 \text{ forbidden.} \end{aligned} \quad \dots\dots\dots(26)$$

Higher order terms in the E1 transition operator can remove the absolute ban on the $T_a = 0$ to $T_b = 0$ transitions. MacDonald (1955) has shown that the effect of these higher order terms is small and that any deviation from the selection rule can be attributed to impurity of the isotopic spin states.

The Coulomb force, being charge dependent, can mix the isotopic spin states and cause a violation of the selection rules. This effect has been studied by Radicati (1953) and by Wilkinson. (1953).

3. Photodisintegration of the Deuteron.

This reaction has been extensively studied both theoretically

and experimentally. The cross-section for the photodisintegration shows a resonance peak near threshold attributable to photomagnetic transitions from the 3S ground state to the virtual 1S state. This theory agrees within experimental error with the measurements of the photodisintegration near threshold.

Above the resonance peak of the photomagnetic reaction, the cross-section exhibits a peak at an energy E_m , twice the threshold energy, $E_m = 4.46$ Mev. The effective range calculations of Bethe and Longmire (1950) and of Austern (1953) give the cross-section in good agreement with experiments for energies up to ~ 25 Mev. For energies greater than ~ 4 Mev, the cross-section is due almost entirely to the electric dipole effect.

The ground state of the deuteron is known to be predominantly a 3S state and hence the most important E 1 transition is the $^3S \rightarrow ^3P$ transition. The transition matrix element is proportional to

$$I = \int \psi_g^* \mathbf{Z} \psi_1 d\tau \quad \dots\dots\dots (27)$$

where ψ_g is the deuteron 3S ground state wave function, ψ_1 is the 3P continuum state wave function and \mathbf{Z} is the \mathbf{Z} component of the inter-nucleon distance \mathbf{r} .

At low energies, there is only a slight interaction in the 3P state of the n-p system and therefore the wave function ψ_1 may be taken as that of the 3P state of the free n-p system. For small energies, this wave function varies as r^2 , so that at these small

energies, the integral I has only a small contribution from values of r inside the nuclear force range. The only effect of the nuclear force range on I is then on the normalisation of ψ_g . This corresponds to the effective range theory as used in neutron-proton and proton-proton scattering.

Thus at moderate energies ψ_g can effectively be replaced by its asymptotic form. The detailed behaviour of the wave function is unimportant. Tensor forces cause a change in the shape of the wave function ψ_g for small values of r , and thus have almost no effect on the cross-section. They do affect the cross-section slightly since the admixture of a 3D state in the deuteron ground state alters the normalisation of the wave function, as well as modifying the form of the effective range theory (Hulthen & Sugawara -1957).

At higher energies transitions become possible from the 3D state to the 3P states by electric dipole radiation. These have been studied theoretically by Austern (1952) but their effect on the cross-section is found to be small.

For energies greater than 10 Mev, the 3P wave function is not zero inside the nuclear force range and the cross-section is reduced. This difference is more important for long-tailed than for short-tailed potentials. Calculations have been made up to 100 Mev for several potentials by Schiff (1950) and by Marshall and Guth (1950).

More recently de Swart and Marshak (1958) have calculated the photodisintegration in the medium energy range (up to 80 Mev) using the Gartenhaus wave function for the deuteron and the Gartenhaus plus

spin orbit wave functions for the final states. These results are in good agreement with experiment and indicate that it is possible to achieve a detailed understanding of the photodisintegration of the deuteron in the medium energy region without renouncing Siegert's theorem or introducing virtual pion effects not contained in Siegert's theorem. It should be noticed that the Gartenhaus wave function used in these calculations, has a larger percentage of D state probability than is normally assumed $\sim 7\%$ as against 2 - 4%.

Zachariasen (1956) has investigated the high energy photodisintegration in the range 100 - 400 Mev. The process is viewed as proceeding by way of the photoproduction of virtual pions followed by meson scattering in deuterium and finally reabsorption by one of the outgoing nucleons. Using the Chew meson theory and the impulse approximation, Zachariasen can explain roughly the qualitative features of the cross-section, notably the secondary resonance at ~ 250 Mev. However the angular distributions do not seem to be fully explained by this model.

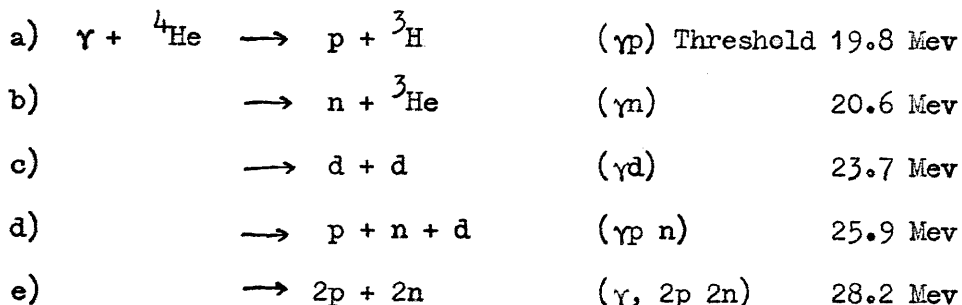
It will be seen that the range of validity of Siegert's theorem is still in doubt but it is reasonable to assume the theorem is valid up to 50 Mev. Further study of this question is being made by de Swart and Marshak.

Thus the photodisintegration of the deuteron can be explained up to ~ 25 Mev without a detailed knowledge of the deuteron wave functions. At higher energies, the details of the wave function become important, and in fact, it is unlikely that the question of the range of validity of Siegert's theorem will be resolved until a detailed wave function

is available.

4. Experimental Data.

The photodisintegration of the α -particle has been studied experimentally by several workers. When the α -particle is bombarded with γ -rays of energy greater than 28 Mev, the following five reactions are possible:-



Contrary to the case of more complicated nuclei, it is possible in the study of the α -particle to analyse the angular distributions fully, and consequently to draw conclusions on the character of photon absorption. This is due to the fact that the final state in the (γn) and (γp) reactions [the fundamental photonuclear reactions in ${}^4\text{He}$] is a two-body system and neither ${}^3\text{H}$ nor ${}^3\text{He}$ have excited states.

The photodisintegration was first studied by Gaerttner and Yeater (1951) who observed the (γp) and (γn) reactions in a cloud chamber with a Bremsstrahlung spectrum of $E_{\gamma \text{ max}} = 100$ Mev. Benedict and Woodward (1951) studied the high energy protons with scintillation counters at an energy of 300 Mev whilst Kikuchi (1952) used nuclear emulsions under similar conditions. Fuller (1954) investigated the (γp) reaction by observing the tracks in nuclear emulsions. In this experiment the plates were placed inside a target chamber containing helium gas, which

was then irradiated with γ -rays of 40 Mev maximum energy. Reid, Swinbank, and Atkinson (1956) have also studied the (γp) reaction using a cloud chamber and a Bremsstrahlung spectrum of maximum energy 330 Mev.

The (γn) reaction has been studied at low energies ($E_{\gamma \text{ max}} = 25$ Mev) by Ferguson et al (1954) and at high energies by de Saussure and Osborne (1955). However considerable doubt has been thrown on the results of the latter paper by the work of Bellamy et al (1957) on the elastic photoproduction of Π^0 mesons in ${}^4\text{He}$. All these experiments were performed over different ranges of energy, in different experimental arrangements, and usually in a way which allowed only one of the reactions (a) - (e) to be observed. The results on the angular distributions and on the energy dependence of the (γp) and (γn) reactions cannot therefore be reliably compared.

This has been remedied recently by the work of Gorbunov and Spiridonov (1958). They investigated the photodisintegration of helium with a cloud chamber in a magnetic field of 5500 gauss. The chamber was irradiated with a Bremsstrahlung spectrum of $E_{\gamma \text{ max}} = 175$ Mev. This method using cloud chambers is the only method allowing the simultaneous observation of all reactions (a) - (e).

Since the coupling between matter and the electromagnetic field is weak ($\frac{e^2}{\hbar c} \ll 1$), the use of perturbation theory and hence of the detailed balance theorem is justified. Thus the results for the inverse process - radiative capture, can be compared with those for the direct process-photodisintegration.

The only inverse reaction to have been studied is



This has been investigated most fully by Perry and Bame (1955) using protons with energies up to 6 Mev.

The results on the photodisintegration can be summarised as follows. These results are taken mainly from the paper of Gorbunov and Spiridonov.

The yields of the various possible photonuclear reactions relative to the (γp) reaction are given in Table I.

I

Reaction	Number of events	Yield relative to (γp)
${}^4\text{He} (\gamma p) {}^3\text{H}$	2835	1
${}^4\text{He} (\gamma n) {}^3\text{He}$	2685	0.95 ± 0.04
${}^4\text{He} (\gamma pn) d +$ ${}^4\text{He} (\gamma 2p 2n) d$	547	0.19 ± 0.01
${}^4\text{He} (\gamma d) d$	≤ 59	≤ 0.02

Within the experimental accuracy the yields for the (γp) and (γn) reactions are the same while the yield of the (γd) reaction does not exceed 2% of the yield of the (γp) reaction. This is expected since both the (γp) and (γn) reactions are allowed in a dipole transition and have practically equal cross-sections (neglecting the slight difference in threshold, and the Coulomb barrier). On the other hand the (γd) reaction is forbidden in a dipole transition because of the isotopic spin selection rules (26).

This is a transition from the state (${}^4\text{He} + \gamma$) with $T = 0$ to the state ($d+d$), also with $T = 0$.

The cross-section for the (γp) reaction is shown in Figure I. It has a maximum of the order of $1.8 \cdot 10^{-27} \text{ cm}^2$ at a photon energy of $\sim 27 \text{ Mev}$.

The integrated cross-section σ_{int} for the (γp) reaction

$$\sigma_{\text{int}}^{(\gamma p)} = \int_0^E \sigma(E) dE$$

is given in Table II.

II

Photon Energy (Mev)	40	100	170
σ_{int}	25.0 ± 1.8	35.8 ± 2.6	37.8 ± 2.8

It should be noted that the energy interval 40 -170 Mev (i.e. the region far removed from the resonance) contributes a considerable fraction $\sim 34\%$ to the integrated cross-section.

The integrated cross-section for photon absorption has also been found.

$$\begin{aligned} \sigma_{\text{int}} &= \int_0^{170} \sigma(E) dE = \sigma_{\text{int}}^{(\gamma p)} + \sigma_{\text{int}}^{(\gamma n)} + \sigma_{\text{int}}^{(\gamma pnd)} \\ &+ \sigma_{\text{int}}^{(\gamma 2p 2n)} + \sigma_{\text{int}}^{(\gamma 2d)} \\ &= 88 \pm 7 \text{ Mev. mb.} \end{aligned} \quad \dots\dots\dots(28)$$

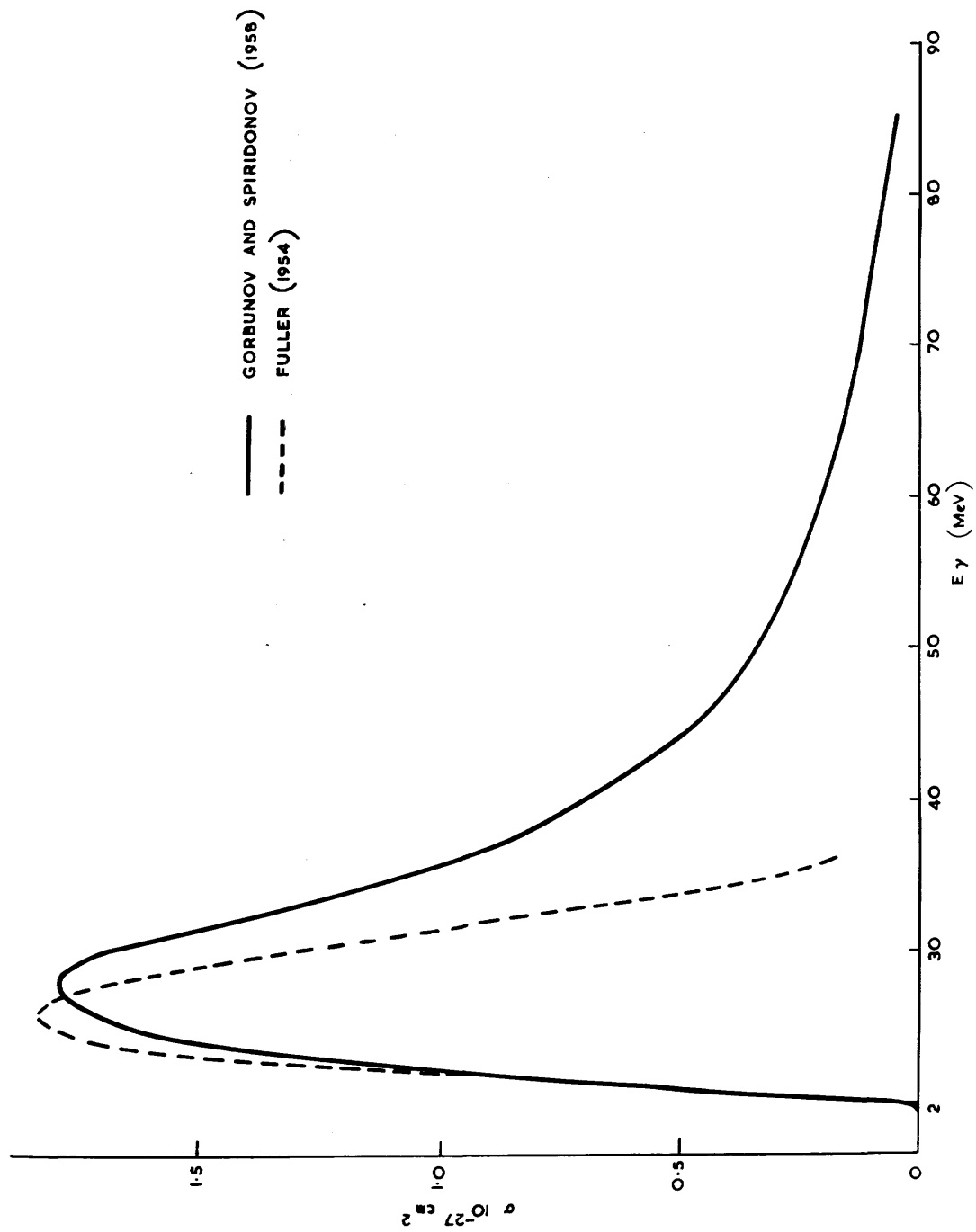


FIGURE 1. EXPERIMENTAL ${}^4\text{He}(\gamma p){}^3\text{H}$ CROSS SECTION

The bremsstrahlung-weighted cross-section was found to be

$$\sigma_b = \int_0^{170} \frac{\sigma(E)}{E} dE = 2.40 \pm 0.15 \text{ mb.} \dots\dots\dots(29)$$

These values are considerably smaller than the values used by Rustgi and Levinger (1957), $\sigma_{int} = 124 \text{ Mev mb.}$ and $\sigma_b = 2.7 \text{ mb.}$ However these authors used the high energy results of de Saussure and Osborne which are now known to be several times too large. This is sufficient to explain the difference.

It is known that the ground state of ^4He is predominantly a 1S_0 state. In Table III are listed the various possible disintegrations of the α -particle caused by E1, M1, and E2 radiation. Also given are the angular distributions in the centre of mass system.

III

Transition	Multipolarity	Proton angular momentum	Spin of $p + ^3\text{H}$	Angular distribution of proton	
$^1S_0 \rightarrow ^1S_0$	X-rays forbidden	$l = 0$	$S = 0$	-	
$^1S_0 \rightarrow ^1P_1$	E1	$l = 1$	$S = 0$	$\sin^2\theta$	Interfere
$^1S_0 \rightarrow ^1D_2$	E2	$l = 2$	$S = 0$	$\sin^2\theta \cos^2\theta$	
$^1S_0 \rightarrow ^3S_1$	M1	$l = 0$	$S = 1$	Isotropic	Interfere
$^1S_0 \rightarrow ^3P_0$	X-rays forbidden	$l = 1$	$S = 1$	-	
$^1S_0 \rightarrow ^3P_1$	E1	$l = 1$	$S = 1$	$1 + \cos^2\theta$	
$^1S_0 \rightarrow ^3D_1$	M1	$l = 2$	$S = 1$	$5 - 3 \cos^2\theta$	
$^1S_0 \rightarrow ^3D_2$	E2	$l = 2$	$S = 1$	$1 - 3 \cos^2\theta + 4 \cos^4\theta$	

The angular distributions are found to be predominantly $\sin^2\theta$ showing that the process takes place mainly by E1 absorption, leaving the proton and triton in a singlet spin state. The experimental results are fitted to the expression

$$A [\sin^2\theta + \beta \sin^2\theta \cos \theta + \gamma \sin^2\theta \cos^2\theta] \dots (30)$$

where A, β , γ are given in Table IV.

IV

E (Mev)	A	β	γ
21 - 30	7.6 ± 0.7	0.11 ± 0.13	-0.2 ± 0.3
30 - 170	6.1 ± 0.6	1.05 ± 0.16	0.53 ± 0.25

As shown in Table III, the term in $\sin^2\theta \cos^2\theta$ corresponds to E2 absorption. Hence below 30 Mev, the (γp) reaction proceeds almost completely by E1 absorption. There occurs a sharp change in the angular distributions at $E_\gamma \approx 30 - 35$ Mev. For energies greater than this, the maximum of the angular distributions has moved forward to an angle of $65 - 70^\circ$. This change shows that from ~ 30 Mev the E2 absorption sets in and produces the interference term $\sin^2\theta \cos \theta$. The analysis indicates that the contribution of E2 absorption to the (γp) cross-section in the interval $E_\gamma = 30 - 170$ Mev, is approximately 10%.

As the region of the resonance, i.e. $E_\gamma \sim 27$ Mev, is of primary

importance in the following calculations, only E1 radiation need be considered to a first approximation.

The photodisintegration of ${}^3\text{He}$ has been studied recently by Cranberg (1958) using a 22 Mev betatron, with nuclear emulsions as detectors. Both the (γp) and (γpn) reactions are observed but it is difficult to separate the data for the two reactions. The cross section for the two-body break-up can be inferred for photon energies in excess of 13 Mev. At 13 Mev it is $1.2 \cdot 10^{-27} \text{ cm}^2$. The (γp) cross section appears to have a maximum for $E_\gamma \sim 9 \text{ Mev}$.

The inverse reaction $D(p, \gamma){}^3\text{He}$ has been studied at low energies by Fowler, Lauritsen and Tollestrup (1949) and by Griffiths and Warren (1955). The former authors studied the reaction with proton energies up to 1.5 Mev. The γ -radiation was found to have predominantly a $\sin^2 \theta$ distribution. The cross-section is given empirically from 0.5 to 1.5 Mev by

$$\sigma = 0.74 E^{0.72} \times 10^{-29} \text{ cm}^2 \quad (E \text{ in Mev}) \dots\dots\dots(31)$$

Griffiths and Warren, using protons with energies 0.3 to 2.0 Mev, found similar results. They found the cross-section for 1 Mev bombarding energy to be $3.2 \pm 0.3 \cdot 10^{-30} \text{ cm}^2$. This is considerably smaller than the value found by Fowler et. al. The difference lies in the normalisation factor, 0.74; the energy dependence is the same.

In addition to the above data on photodisintegration and radiative

capture, the root-mean-square (r.m.s.) radii R of the lightest nuclei have recently been determined accurately by the electron scattering experiments of Hofstadter (1956) and McAllister (1956). The r.m.s. radii of the proton, deuteron, and α -particle are

$$R_p = 0.78 \cdot 10^{-13} \text{ cm.} \quad R_D = 1.96 \cdot 10^{-13} \text{ cm.}$$

$$R_\alpha = 1.61 \cdot 10^{-13} \text{ cm.}$$

It can be shown that R is related to 6_p , assuming simple symmetry properties for the nuclear wave function. This will be considered in detail later.

To complete the data on the lightest nuclei the binding energies are given below.

$$\text{BE } (^2\text{H}) = 2.226 \text{ Mev.} \quad \text{BE } (^3\text{He}) = 7.72 \text{ Mev.}$$

$$\text{BE } (^3\text{H}) = 8.49 \text{ Mev.} \quad \text{BE } (^4\text{He}) = 28.2 \text{ Mev.}$$

The Coulomb energy of ^3He is 0.77 Mev.

5. Electric Dipole Transitions.

The matrix element for an E1 transition in the "long-wavelength" approximation is given by

$$\begin{aligned} Q_{10}^{(0)} &= e \sum_{i=1}^Z \int r_i Y_{10}(\theta_i \phi_i) \phi_a^x \phi_b d\tau \\ &= \sqrt{\frac{3}{4\pi}} e \sum_{i=1}^Z \int \phi_a^x Z_i \phi_b d\tau \quad \dots\dots\dots (32) \end{aligned}$$

where Z is the atomic number, the sum being over all protons.

In the following the matrix element M_{ab} will be studied, where

$$M_{ab} = \sum_{i=1}^Z \int \phi_a^* \mathbf{r}_i \phi_b d\tau \quad \dots\dots\dots (33)$$

and \mathbf{r}_i is the position vector of the i^{th} proton relative to the centre of mass.

To evaluate the matrix element (33) both ϕ_a and ϕ_b must be available. As very little is known about the form of the ground state wave function ϕ_a and even less about the wave functions ϕ_b of excited states, explicit assumptions must be made, as to the form of ϕ_a and ϕ_b .

Flowers and Mandl (1951) and Gunn and Irving (1951) have evaluated the matrix element in the Born approximation, and find that they can account for the qualitative features of the (γp) cross-section without assuming the existence of an excited state of the α -particle.

However, the method of Levinger and Bethe (1950) can be used, summing over all final states ϕ_b and applying closure to the matrix elements. The results then depend only on the ground state wave function ϕ_a . This sum rule method leads to expressions for the integrated cross-section for photon absorption.

$$\sigma_{\text{int}} = \int_0^\infty \sigma(E) dE \quad \dots\dots\dots (34)$$

and the bremsstrahlung-weighted cross-section

$$\sigma_b = \int_0^\infty \frac{\sigma(E)}{E} dE \quad \dots\dots\dots (35)$$

Here $\sigma(E)$ is the sum of all partial cross-sections for the various nuclear reactions that may occur subsequent to electric dipole photon absorption:- (γp) , (γn) , $(\gamma \gamma^1)$ etc.

Levinger and Bethe (1950) found that for a degenerate Fermi gas model of the nucleus

$$\sigma_{int} = \frac{2\pi^2 e^2 \hbar}{m c} \frac{N Z}{A} (1 + 0.8x) \dots\dots\dots(36)$$

where Z is the number of protons, N the number of neutrons and $A = N + Z$. x is the fraction of Majorana exchange force present in the neutron-proton force. Hence the fraction of exchange force can be estimated by comparing this result with experiment. The result is dependant on the nuclear model assumed only through the numerical factor 0.8 in (36).

In the following, both these approaches will be considered in turn.

Shortly after this work was completed, a paper by Levinger and Rustgi (1957) appeared in which similar formulae were deduced for the integrated cross-section for photon absorption in ^4He .

Before giving details of the calculations, the wave functions to be used for the α -particle must be considered.

6. Alpha-Particle Wave Functions.

Simple analytic wave functions for the three and four particle nuclei have been used by several authors whilst considering the "consistency" problem of light nuclei. Briefly stated, the consistency problem is an attempt to find a phenomenological two-body potential with suitable parameters, which yields the experimental binding energies of ^2H , ^3H , ^3He and ^4He , the deuteron quadrupole moment, and which is consistent with the low energy two-body scattering data.

The earliest calculations were carried out using a central interaction. This could account for the binding energies of ^2H and ^3H but gave too large a value for the ^4He binding energy and also for the Coulomb energy of ^3He . Central forces fail, also, to explain the deuteron quadrupole moment.

The next stage was to consider a mixture of central and tensor forces. In the "consistency" problem it is usual to consider a potential of the form

$$V(\underline{r}_{12}) = -V_0 \left[1 + \frac{1}{2}g(\underline{\sigma}_1 \cdot \underline{\sigma}_2 - 1) \right] J_C \left(\frac{r}{r_c} \right) - \gamma V_0 S_{12} J_T \left(\frac{r}{r_t} \right) \dots\dots\dots (37)$$

where $r = |\underline{r}_1 - \underline{r}_2|$, $\underline{\sigma}_1, \underline{\sigma}_2$ are the Pauli spin operators and S_{12} is the usual tensor operator

$$S_{12} = \frac{3(\underline{\sigma}_1 \cdot \underline{r})(\underline{\sigma}_2 \cdot \underline{r})}{r^2} - (\underline{\sigma}_1 \cdot \underline{\sigma}_2) \dots\dots\dots (38)$$

Extensive sets of values for the parameters V_0, g, γ, r_c, r_t which satisfy the deuteron binding energy and quadrupole moment, have been tabulated by Feshbach and Schwinger (1951). These authors used a Yukawa well

$$J_C(x) = J_T(x) = \frac{e^{-x}}{x} \dots\dots\dots (39)$$

for both central and tensor forces, but assumed different ranges r_c and r_t .

For the three and four particle nuclei, the spatial wave

functions considered, have generally been functions of the variable ρ where ρ is given by

$$\rho^2 = \sum_{\substack{i,j=1 \\ i < j}}^A r_{ij}^2 ; \quad \underline{r}_{ij} = \underline{r}_i - \underline{r}_j \quad \dots\dots\dots (40)$$

The forms considered so far, are

(i)	Gaussian	$\phi^G = \exp - \mu \rho^2$	
(ii)	Exponential	$\phi^E = \exp - \mu \rho$	$\dots\dots\dots (41)$
(iii)	Irving	$\phi^I = \frac{\exp - \mu \rho}{\rho}$	

Also considered have been functions of the variable, given by

$$\eta = \sum_{\substack{i,j=1 \\ i < j}}^A \frac{1}{r_{ij}} \quad \dots\dots\dots (42)$$

The form most often used is

(iv)	Feshbach	$\phi^F = \exp - \mu \eta$	$\dots\dots\dots (41)$
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Assuming that central forces predominate the ground state of the α -particle is an S-state. In the second approximation, D-states must be introduced since the tensor force couples S- and D- states directly but not S- and P- states. Irving (1953) has considered the effect of the tensor force on the α -particle binding energy. He used the form (41(ii)) for the spatial wave functions. The non-radial parts were obtained using the operator formalism introduced by Gergouy and Schwinger (1942). Only the principal D-state was considered, but the calculations have recently been extended by Abraham et al (1955) to include six

D- states. These authors find that only three of these make any significant contribution to the binding energy. In both these calculations, the standard variational method was used to determine the values of the parameters μ_1 which minimised the energy. Abraham et al conclude that it may be possible to satisfy the consistency problem using the potential (37,39) with suitable parameters.

Spatial wave functions of the Feshbach type (41 iv) have been used successfully in calculations on the binding energy of the triton by Pease and Feshbach (1951)(1952). In fact, Irving (1953) finds that this form gives better results for the triton than the Exponential form (41 ii). However, for the α -particle, the only calculation using the Feshbach form is that of Frohlich et al (1947). Their calculations using central forces had to be carried out numerically and were extremely tedious. Owing to this mathematical complexity, no further use has been made of this form of wave function for the α -particle.

At this point another calculation on the binding energy of ^4He must be mentioned. Clark (1954) using a method introduced by Morpurgo (1952), has considered the ground state as an admixture of an S- state and two D- states, using the same formalism as Irving. Clark introduced a function $\gamma(\rho)$ with no adjustable parameters, the form of the function γ itself being variable. He found that γ must satisfy a second order differential equation of the one-dimensional Schroedinger type. He solved this equation approximately to give an analytic form for the wave function. This wave function gave a

binding energy of 29.6 Mev which is slightly greater than the experimental one (28.2 Mev).

In all these calculations, only spatially symmetric wave functions have been used. This is justified since it is known that the ground states of the three and four particle nuclei consist largely of antisymmetric spin states. For example, Brown (1939) in a variational calculation obtained a 4% admixture of symmetric spin function in the case of ${}^3\text{H}$. For ${}^4\text{He}$, this percentage would not be expected to be greater and it is probably less.

The wave functions discussed above will be used in various calculations on the photodisintegration of the α -particle. Since the theory of photodisintegration is reasonably well understood, this will serve to test the adequacy of the various wave functions and hence of the internucleon potentials which were used to derive them.

7. Previous and Present Calculations.

Experiment has shown that photon absorption by the α -particle is dominantly electric dipole with a small admixture of electric quadrupole, as was expected from general multipole radiation theory.

This is borne out by the explicit calculations of Flowers and Mandl (1951). These authors find that the E1 cross section is about one hundred times the E2 cross section in the region of the experimental resonance. Also they found that M2 transitions take place only through states with antisymmetric spatial wave functions and hence their probability is negligible.

In previous calculations of the matrix elements, the central force approximation has been used. Flowers and Mandl used Gaussian type wave functions whilst Gunn and Irving (1951) used those of the Gaussian and Irving types. Both of these calculations used the Born approximation i.e. the emitted nucleon is described by a plane wave. This approximation will be justified in the following. The calculations are extended to include the Exponential type wave function and also to take the effect of the tensor force into account.

The sum rule method had been applied to the α -particle using the wave functions discussed in the last section (66), notably those of Abraham et. al. and of Clark. These calculations are compared with the work of Levinger and Rustgi (1957) who used the Exponential form with parameters due to Irving (1953).

In addition to the above calculations, the photodisintegration of ${}^4\text{He}$ has been considered by Gamba (1951, 1952) using a group theoretical approach. The photodisintegration of the three body nuclei has been studied by Verde (1950) using the symmetry properties of the nuclear wave function. However, these authors consider only central forces and it is difficult to extend their calculations to include the effect of the tensor force. It is unlikely that further progress will be made using this approach and it will not be considered further here.

CHAPTER II

Sum Rule Calculations

1. Electric Dipole Transition Operator.

It is seen from (I.33) that the electric dipole transition operator Q_1 (allowing for the recoil of the centre of mass) is given by

$$Q_1 = \sum_p e_p (\underline{r}_p - \underline{R}_A) \quad \dots\dots\dots (1)$$

where the sum is over all protons.

The centre of mass coordinates of the nucleons, protons and neutrons respectively are defined by

$$\underline{R}_A = \frac{1}{A} \sum_{i=1}^A \underline{r}_i \quad \underline{R}_p = \frac{1}{Z} \sum_{p=1}^Z \underline{r}_p \quad \underline{R}_N = \frac{1}{N} \sum_{n=1}^N \underline{r}_n \quad \dots\dots\dots (2)$$

where Z is the number of protons in the nucleus, N the number of neutrons and $A = N + Z$, the atomic **mass number**

Then the E1 transition operator is

$$Q_1 = e \sum_p (\underline{r}_p - \underline{R}_A) = e \frac{N}{A} \sum_p \underline{r}_p - e \frac{Z}{A} \sum_n \underline{r}_n \quad \dots\dots\dots (3)$$

$$= e \frac{NZ}{A} (\underline{R}_p - \underline{R}_N) = \frac{eNZ}{A} \underline{R}_{pN} \quad \dots\dots\dots (4)$$

where $\underline{R}_{pN} = \underline{R}_p - \underline{R}_N$, the coordinate of the centre of mass of the protons relative to the centre of mass of the neutrons.

Written in the form (3), Q_1 is the sum of single particle operators and connects wave functions differing at most by the state of one particle. The protons have effective charge $\left(e \frac{N}{A}\right)$ and the neutrons $\left(-e \frac{Z}{A}\right)$.

The individual particle model for the nuclear photoeffect has been developed particularly by Wilkinson (1956) and Rand (1957). An individual nucleon in the nucleus absorbs the incoming photon, and, for an E1 transition, is excited to the next shell of opposite parity. Since, for reasonable shell model potentials, all possible transitions have approximately the same energy, there should be a peak in the absorption cross section at this energy. The finite width of the peak arises partly because of the slight spread in single particle absorption energies and partly because the single particle states decay by inelastic collisions in the nucleus (cloudy crystal ball effect).

Written in the form (4), Q_1 is expressed in terms of a collective coordinate R_{pn} . This has led to the collective model of Goldhaber and Teller (1948) and of various other authors, for photon absorption. The width of the giant resonance reflects the damping of the collective mode due to viscous effects arising from the relative motion of the proton and neutron 'clouds'.

It seems probable that the wave function of the state reached in the dipole photoeffect should be thought of as a mixture of shell model and collective model wave functions. This subject has been

studied very recently by Levinger (1958).

The following calculations have the merit of being model-independent since the final state does not occur explicitly in the results.

It is convenient, however, to use the form (4) for Q_1 and to introduce the general coordinate system (1) of Appendix A.

2. Sum Rules. General Formalism.

The electric dipole transition matrix element M_{ab} of (I.33) is given by

$$M_{ab} = \frac{NZ}{A} \int \psi_b^* (R_{pN})_z \psi_a d\tau \quad \dots\dots\dots (5)$$

for a transition between the ground state ψ_a and the final state ψ_b . $\int d\tau$ implies integration over all position coordinates and summation over all spin coordinates.

In the following the sum over all final states ψ_b will be taken and using the closure relation for two operators A , B

$$\sum_b \langle a | A | b \rangle \langle b | B | a \rangle = \langle a | AB | a \rangle \quad \dots\dots\dots (6)$$

it is seen that the results will depend only on the wave function assumed for the ground state. Here \sum_b implies a sum over the discrete levels and integration over the continuum, and

$$\langle a | A | b \rangle = \int \psi_a^* A \psi_b d\tau$$

The oscillator strength for the above transition is defined as

$$f_{ab} = \frac{2m}{\hbar^2} (E_b - E_a) |M_{ab}|^2 \dots\dots\dots (7)$$

The cross section for the electric dipole absorption of a photon of energy $E = E_b - E_a$ is proportional to the oscillator strength

$$\sigma_{ab} = \frac{2\pi^2 e^2 \hbar}{mc} f_{ab} \dots\dots\dots (8)$$

The method of Levinger and Bethe (1950) will be followed in evaluating the various moments of the summed oscillator strength

$$\sum_b f_{ab} (E_b - E_a)^S \dots\dots\dots (9)$$

These moments correspond to

$$\int E^S \sigma(E) dE = \frac{2\pi^2 e^2 \hbar}{mc} \sum_b f_{ab} (E_b - E_a)^S \dots\dots\dots (10)$$

Of greatest interest are (a) the integrated cross section

$$\sigma_{int} = \int \sigma(E) dE, \quad S = 0 \dots\dots\dots)10a)$$

and (b) the bremsstrahlung - weighted cross section

$$\sigma_b = \int \frac{\sigma(E)}{E} dE, \quad S = -1 \dots\dots\dots (10b)$$

The Schrodinger equations for ψ_a and ψ_b^* are

$$\begin{aligned} (T + V - E_a) \psi_a &= 0 \\ (T + V - E_b) \psi_b^* &= 0 \end{aligned} \dots\dots\dots (11)$$

where T is the kinetic energy operator

$$T = -\frac{\hbar^2}{2m} \sum_{i=1}^A \nabla_i^2 \quad \dots\dots\dots (12)$$

and V the potential energy operator

$$V = \sum_{\substack{i,j=1 \\ i < j}}^A V(ij) \quad \dots\dots\dots (13)$$

From the equations (11) for χ_a and χ_b^* it is readily seen that

$$\begin{aligned} (E_b - E_a) \chi_b^* (R_{pN})_Z \chi_a \\ = \chi_a (R_{pN})_Z (T+V) \chi_b^* - \chi_b^* (R_{pN})_Z (T+V) \chi_a \end{aligned}$$

Hence

$$\begin{aligned} (E_b - E_a) M_{ab} &= \frac{NZ}{A} \int \left\{ \chi_a (R_{pN})_Z T \chi_b^* - \chi_b^* (R_{pN})_Z T \chi_a \right\} d\tau \\ &+ \frac{NZ}{A} \int \left\{ \chi_a (R_{pN})_Z V \chi_b^* - \chi_b^* (R_{pN})_Z V \chi_a \right\} d\tau \\ &= \frac{NZ}{A} \left\{ I_T + I_V \right\} \quad \dots\dots\dots (14) \end{aligned}$$

The integral involving the kinetic energy, I_T , and that involving the potential energy I_V will be considered separately.

$$I_T = \int \left\{ \chi_a (R_{pN})_Z T \chi_b^* - \chi_b^* (R_{pN})_Z T \chi_a \right\} d\tau \quad \dots\dots\dots (15)$$

It is shown in Appendix A that

$$T = -\frac{\hbar^2}{2m} \sum_i \nabla_i^2$$

$$T = -\frac{\hbar^2}{2m} \left\{ \frac{1}{A} \nabla_{R_A}^2 + \frac{A}{N\hbar} \nabla_{R_{pN}}^2 + \sum_{l=1}^{Z-1} \left(1 + \frac{1}{l}\right) \nabla_{R_{lp}}^2 + \sum_{l=1}^{N-1} \left(1 + \frac{1}{l}\right) \nabla_{R_{ln}}^2 \right\}$$

By integrating by parts, it is easily shown that

$$\int \psi_a(R_{pN}) \nabla_{\underline{u}}^2 \psi_b^* d\tau = \int \psi_b^*(R_{pN}) \nabla_{\underline{u}}^2 \psi_a d\tau, \quad \underline{u} = R_A, R_{lp}, R_{ln}$$

$$\int \psi_a(R_{pN}) \nabla_{R_{pN}}^2 \psi_b^* d\tau = \int \psi_b^*(R_{pN}) \nabla_{R_{pN}}^2 \psi_a d\tau + 2 \int \psi_b^* \frac{d}{dR_{pN}} \psi_a d\tau$$

Hence the kinetic energy integral is

$$I_T = -\frac{\hbar^2}{m} \frac{A}{N\hbar} \int \psi_b^* \frac{d}{dR_{pN}} \psi_a d\tau \quad \dots\dots\dots (16)$$

The potential energy integral I_V is given by

$$I_V = \int \left\{ \psi_a(R_{pN}) \nabla \psi_b^* - \psi_b^*(R_{pN}) \nabla \psi_a \right\} d\tau \quad \dots\dots\dots (17)$$

It should be noticed here that I_V is zero if the potential V commutes with R_{pN} . Hence contributions are to be expected only from space exchange forces and from velocity dependent forces. Spin exchange forces do not contribute. A general two-body potential of the exchange type containing both central and tensor forces will be chosen. Velocity dependent forces such as the spin orbit force, will not be considered. The two-body potential can be written as (Blatt and Weisskopf p 139)

$$V(ij) = (w + m M_{ij} + b B_{ij} + h M_{ij} B_{ij}) V_c(ij) + (w^1 + m^1 M_{ij}) S_{ij} V_T(ij) \quad \dots\dots\dots (18)$$

where w, m, b, h, w^1, m^1 are constants.

M_{ij} is the Majorana space exchange operator.

B_{ij} is the Bartlett spin exchange operator.

S_{ij} is the usual tensor operator.

$$S_{ij} = \frac{3 (\underline{g}_i \cdot \underline{r}_{ij}) (\underline{g}_j \cdot \underline{r}_{ij})}{r_{ij}^2} - (\underline{g}_i \cdot \underline{g}_j)$$

and $V_c(ij)$ and $V_T(ij)$ are the radial forms of the central and tensor potentials respectively.

Now since $(R_{pN})_Z$ commutes w, m, b, \hbar , w^1 , m^1 , B_{ij} , and S_{ij}

$$\begin{aligned} & V(ij) (R_{pN})_Z - (R_{pN})_Z V(ij) \\ &= \left\{ (m + \hbar B_{ij}) V_c(ij) + m^1 S_{ij} V_T(ij) \right\} \left\{ M_{ij} (R_{pN})_Z - (R_{pN})_Z M_{ij} \right\} \end{aligned}$$

From the form of R_{pN} (4), it is seen that if i and j both denote protons or both neutrons, then M_{ij} and $(R_{pN})_Z$ commute.

Hence

$$\begin{aligned} & \sum_{ij} \left\{ V(ij) (R_{pN})_Z - (R_{pN})_Z V(ij) \right\} \\ &= \sum_{p,n} \left\{ (m + \hbar B_{pn}) V_c(pn) + m^1 S_{pn} V_T(pn) \right\} \left\{ M_{pn} (R_{pN})_Z - (R_{pN})_Z M_{pn} \right\} \end{aligned}$$

where the sum is over all neutron-proton pairs.

In a potential of the form (18), only the space exchange terms i.e. the Majorana and the Heisenberg terms contribute. Now,

$$M_{pn} (R_{pN})_Z - (R_{pN})_Z M_{pn} = -\frac{A}{NZ} (r_p - r_n)_Z M_{pn} \dots\dots\dots (19)$$

and hence

$$I_V = -\frac{A}{NZ} \int \psi_b^* \sum_{p,n} \left\{ (m + h B_{pn}) V_c(pn) + m^1 S_{pn} V_T(pn) \right\} \psi_{pn} M_{pn} \psi_a d\tau \dots\dots\dots (20)$$

where $\mathbf{r}_{pn} = (\mathbf{r}_p - \mathbf{r}_n)$

Therefore, gathering the results together gives

$$M_{ab} = \frac{NZ}{A} \int \psi_b^* (R_{pn}) \psi_a d\tau \dots\dots\dots (5)$$

and

$$\begin{aligned} (E_b - E_a) M_{ab} &= \frac{NZ}{A} \{ I_T + I_V \} \dots\dots\dots (14) \\ &= -\frac{\hbar^2}{m} \int \psi_b^* \frac{d}{dR_{pn}} \psi_a d\tau - \int \psi_b^* \sum_{pn} U(pn) \psi_{pn} M_{pn} \psi_a d\tau \dots\dots\dots (21) \end{aligned}$$

where

$$U(pn) = (m + h B_{pn}) V_c(pn) + m^1 S_{pn} V_T(pn) \dots\dots\dots (22)$$

$$= U_c(pn) + U_T(pn) \dots\dots\dots (23)$$

The sum rules are obtained from (5) and (21) by summing over all final states ψ_b and applying the closure relation (6).

$$\begin{aligned} (1) \quad \sum_p \mathcal{B}_{ab} / (E_b - E_a) &= \frac{2m}{\hbar^2} \sum_b |M_{ab}|^2 \\ &= \frac{2m}{\hbar^2} \left(\frac{NZ}{A} \right)^2 \int \psi_a^* (R_{pn})^2 \psi_a d\tau \\ &= \frac{2m}{3\hbar^2} \left(\frac{NZ}{A} \right)^2 \int \psi_a^* (R_{pn})^2 \psi_a d\tau \dots\dots\dots (24) \end{aligned}$$

$$\begin{aligned}
 (2) \quad \sum_b f_{ab} &= \frac{2m}{\hbar^2} \sum_b (E_b - E_a) |M_{ab}|^2 \\
 &= \frac{2m}{\hbar^2} \sum_b (E_b - E_a) M_{ba} \cdot M_{ab}
 \end{aligned}$$

It is more convenient to rewrite this in the symmetric form

$$\begin{aligned}
 \sum_b f_{ab} &= \frac{m}{\hbar^2} \sum_b \left\{ M_{ba} [(E_b - E_a) M_{ab}] - [(E_a - E_b) M_{ba}] M_{ab} \right\} \\
 &= \frac{m}{\hbar^2} \frac{NZ}{A} \left[-\frac{\hbar^2}{m} \int \psi_a^x \left\{ (R_{pn})_Z \frac{d}{dR_{pnZ}} - \frac{d}{dR_{pnZ}} (R_{pn})_Z \right\} \psi_a d\tau \right. \\
 &\quad \left. - \int \psi_a^x \sum_{pn} U(pn) \psi_{pn} \left\{ (R_{pn})_Z M_{pn} - M_{pn} (R_{pn})_Z \right\} \psi_a d\tau \right] \\
 &= \frac{NZ}{A} \int \psi_a^x \psi_a d\tau - \frac{NZ}{A} \frac{m}{\hbar^2} \int \psi_a^x \sum_{pn} U(pn) \psi_{pn} \left[(R_{pn})_Z M_{pn} \right] \psi_a d\tau
 \end{aligned}$$

where the square brackets denote the commutator, and finally, by using (19).

$$\sum_b f_{ab} = \frac{NZ}{A} - \frac{m}{\hbar^2} \int \psi_a^x \sum_{pn} U(pn) \psi_{pn}^2 M_{pn} \psi_a d\tau \quad \dots\dots\dots (25)$$

This is the most symmetrical form for $\sum_b f_{ab}$. However for explicit calculation, it is convenient to simplify this form further.

By the Pauli Exclusion Principle, the wave function is antisymmetric under the exchange of space and spin coordinates of any pair of like nucleons. Hence the sum occurring in the second term of (25) may be reduced to a single term by interchanging the proton p with the proton P (say) and interchanging the neutron n with the Neutron N. Hence

$$\sum_b f_{ab} = \frac{NZ}{A} - \frac{m}{\hbar^2} NZ \int \psi_a^x U(PN) \psi_{PN}^2 M_{PN} \psi_a d\tau \quad \dots\dots\dots (26)$$

It is of interest at this point to compare the result (25) for the electric dipole oscillator sum with that of previous authors.

The form (25) is the same as Equation (LB11) of Levinger and Bethe (1950) but the centre of mass motion has been introduced in a more natural form. Also (25) is exact whereas, in deriving (LB11), the assumption of similar proton and neutron wave functions, was used to remove correlation terms (Levinger 1956). This approximation procedure is seen to be unnecessary.

Sachs and Austern (1951) have derived general sum rules for electric multipole radiation from the requirements of gauge invariance. They found

$$\sum_b f_{ab} = -\frac{m}{\hbar^2} \langle t_a | [[H, D_1], D_1] | t_a \rangle \dots\dots\dots (27)$$

where D_1 is the electric 2^1 - pole operator and H is the Hamiltonian. This result can be derived from the Schroedinger equation by the method used above without explicitly using the requirements of gauge invariance. (See Appendix B). The properties of gauge invariance were used by Sachs and Austern to show that D_1 is independent of the form of the interaction.

Levinger and Bethe considered only Majorana exchange forces. This has been extended in the above derivation to include a general exchange mixture and also tensor forces. This extension has also been made by Levinger and Rustgi (1957) in a paper which appeared after this work had been completed. However, there are some errors in their paper. These will be discussed later.

The inclusion of spin orbit forces (neglected above) and other velocity dependent forces, has been considered by Frankel (1955) and by Levinger, Austern, and Morrison (1957).

If the Hamiltonian in (27) is written as $H = T + V$, the kinetic term involves the momenta of all the particles quadratically and leads to the usual Thomas - Reiche - Kuhn result for electrons.

For a nuclear system, in which the Hamiltonian contains the coordinates and momenta of N neutrons and Z protons but those of no other particles (such as mesons), the kinetic term leads to a contribution $\frac{NZ}{A}$ and (27) reduces to

$$\sum_b f_{ab} = \frac{NZ}{A} + \sum_b^1 f_{ab} \dots\dots\dots (28)$$

The second term depends on the potential energy operator and (28) is equivalent to (25).

The sum rule for $\sum_b f_{ab} / (E_b - E_a)$ is discussed in the next section, §3.

3. Root Mean Square Radius. - Definition.

The root - mean - square (r.m.s.) radius of a nucleus will now be defined.

The r.m.s. radius of the mass distribution is defined as

$$R_m^2 = \frac{1}{A} \sum_{i=1}^A \int \tau^x (\underline{r}_i - \underline{R}_A)^2 \tau_o d\tau \dots\dots\dots (29)$$

and of the charge distribution as

$$R_c^2 = \frac{1}{Z} \sum_{i=1}^Z \int \tau^x (\underline{r}_i - \underline{R}_A)^2 \tau_o d\tau \dots\dots\dots (30)$$

where ψ_0 is the ground state wave function.

It is readily shown that

$$R_m^2 = \frac{1}{A^2} \int \psi_0^* \rho_A^2 \psi_0 d\tau \quad \dots\dots\dots (31)$$

where ρ_A^2 is defined as in Appendix (A.3).

$$\rho_A^2 = \sum_{\substack{i,j=1 \\ i < j}}^A \underline{r}_{ij}^2 \quad \dots\dots\dots (32)$$

The most accurate measurements of the mean square radius of the charge distribution are those obtained by high energy electron scattering (Hofstadter 1956). However, care must be exercised in comparing the experimental results with the above definitions since the experiments have shown that the proton has a finite charge distribution $R_p = 0.77 \cdot 10^{-13}$ cm. The above definition (30) holds only for a proton with a point charge. Following the treatment of Foldy (1957) and others, the correction for the finite charge distribution of the proton is made by using

$$R_c^2 (\text{exp}) = R_c^2 (\text{theor}) + R_p^2 \quad \dots\dots\dots (33)$$

where $R_c (\text{exp})$ and $R_c (\text{theor})$ are the experimental and theoretical values respectively.

The bremsstrahlung - weighted cross-section is given by (10b) and (24)

$$\zeta_b = \frac{2 \pi^2 e^2 \hbar}{mc} \sum_b f_{ab} / (E_b - E_a)$$

$$= \frac{4\pi^2}{3} \frac{e^2}{\hbar c} \left(\frac{NZ}{A} \right)^2 \langle R_{pN}^2 \rangle_{\infty} \dots\dots\dots (34)$$

It will now be shown that ϵ_p is closely related to R_m and R_c .

From the definition (4) of R_{pN}

$$\begin{aligned} \langle R_{pN}^2 \rangle_{\infty} = & \langle \frac{1}{Z^2} \sum_p r_p^2 + \frac{1}{N^2} \sum_n r_n^2 + \frac{1}{Z^2} \sum_{p \neq p'} r_p r_{p'} + \frac{1}{N^2} \sum_{n \neq n'} r_n r_{n'} \\ & - \frac{2}{NZ} \sum_{pn} r_p r_n \rangle_{\infty} \end{aligned}$$

Now, by the Pauli Exclusion Principle, the wave function is antisymmetric under the exchange of the space and spin coordinates of any pair of like nucleons. Since R_{pN}^2 does not involve the spin operators, it is seen that $\langle r_p^2 \rangle$ has the same value α_p for all protons, $\langle r_n^2 \rangle$ has the same value α_N for all neutrons, $\langle r_p r_{p'} \rangle$ has the same value β_{pp} for all distinct pairs of protons, $\langle r_n r_{n'} \rangle$ has the same value β_{NN} for all distinct pairs of neutrons, and $\langle r_p r_n \rangle$ has the same value β_{pN} for all neutron proton pairs. Hence

$$\langle R_{pN}^2 \rangle_{\infty} = \frac{1}{Z} \alpha_p + \frac{1}{N} \alpha_N + \frac{Z-1}{Z} \beta_{pp} + \frac{N-1}{N} \beta_{NN} - 2\beta_{pN} \dots\dots\dots (35)$$

Similarly it can be shown that

$$\begin{aligned} R_c^2 = & \left\{ 1 - \frac{2}{A} + \frac{Z}{A^2} \right\} \alpha_p + \frac{N}{A^2} \alpha_N - \frac{(2A-Z)(Z-1)}{A^2} \beta_{pp} \\ & + \frac{N(N-1)}{A^2} \beta_{NN} - \frac{2NZ}{A^2} \beta_{pN} \dots\dots\dots (36) \end{aligned}$$

and

$$R_m^2 = \frac{Z(A-1)}{A^2} \alpha_p + \frac{N(A-1)}{A^2} \alpha_N - \frac{Z(Z-1)}{A^2} \beta_{pp} +$$

$$+ \frac{N(N-1)}{A^2} \beta_{NN} - \frac{2NZ}{A^2} \beta_{pN} \dots\dots\dots (37)$$

The formulae (35) (36) (37) are exact.

For the lightest nuclei, they may be simplified further by the use of the symmetry properties of the nuclear wave function.

(1) If the ground state wave function is symmetric under the interchange of all protons with all neutrons, then for a self-conjugate nucleus

$$N = Z = \frac{A}{2} ; \alpha_p = \alpha_N = \alpha ; \beta_{pp} = \beta_{NN} = \beta ; \beta_{pN} = \gamma \dots\dots\dots (38)$$

It is shown easily that

$$R_c^2 = \frac{A-1}{A} \alpha - \frac{(Z-1)}{2Z} \beta - \frac{2Z^2}{A^2} \gamma = R_m^2 \dots\dots\dots (39)$$

(2) If the ground state wave function is completely symmetric in the space coordinates of all nucleons, then

$$\alpha_p = \alpha_N = \alpha ; \beta_{pp} = \beta_{pN} = \beta_{NN} = \beta \dots\dots\dots (40)$$

giving

$$\langle R_{pN}^2 \rangle_{\infty} = \frac{A}{NZ} (\alpha - \beta) \dots\dots\dots (41)$$

$$R_c^2 = R_m^2 = \frac{A-1}{A} (\alpha - \beta) \dots\dots\dots (42)$$

Hence

$$G_b = \frac{4\pi^2}{3} \frac{e^2}{\hbar c} \frac{NZ}{A-1} R^2 \dots\dots\dots (43)$$

This relation has also been derived by Foldy (1957) and his notation has been used here.

The justification of the assumption (40) will be considered explicitly in the case of the α -particle, but first wave functions for the α -particle must be introduced.

4. Alpha-Particle Wave Functions.

As discussed in (I§6.), the wave functions used by previous authors for the α -particle have generally been functions of ρ where ρ is given by

$$\rho^2 = \sum_{\substack{i,j=1 \\ i < j}}^4 r_{ij}^2 \quad \dots\dots\dots (44)$$

For convenience the coordinate system (A7) will be used

$$\underline{u} = \underline{r}_2 - \underline{r}_1 \quad \underline{v} = \underline{r}_4 - \underline{r}_3 \quad \dots\dots\dots (45)$$

$$\underline{R} = \frac{1}{2} (\underline{r}_3 + \underline{r}_4) - \frac{1}{2} (\underline{r}_1 + \underline{r}_2)$$

$$\underline{R}_4 = \frac{1}{4} (\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4)$$

where 1,2 denote neutrons and 3,4 protons.

The calculations of Abraham et al (1955) on the binding energy of the α -particle were performed by considering the ground state as a mixture of an S- state and six D-states. These authors found that only three of these D-states gave any significant contribution to the binding energy and hence only these three D-states will be considered here.

The α -particle wave function is taken to be of the form

$$\begin{aligned} \psi_\alpha = a_s \psi_s + a_1 \psi_1 + a_2 \psi_2 + a_3 \psi_3 \\ \dots\dots\dots (46) \end{aligned}$$

where

$$\begin{aligned} \psi_s = N_s \phi_s(\rho) |s\rangle \quad \psi_2 = N_2 \phi_2(\rho) |d_2\rangle \\ \dots\dots\dots (46a) \end{aligned}$$

$$\psi_1 = N_1 \phi_1(\rho) |d_1\rangle \quad \psi_3 = N_3 \phi_3(\rho) |d_3\rangle$$

and the spin-angle functions are obtained by the operator method introduced by Gerjuoy and Schwinger (1942).

$$|s\rangle = \frac{1}{2} \left\{ \alpha(1) \beta(2) - \beta(1) \alpha(2) \right\} \left\{ \alpha(3) \beta(4) - \beta(3) \alpha(4) \right\}$$

$$|d_1\rangle = |\underline{u}, \underline{y}\rangle$$

$$|d_2\rangle = (\underline{y} \cdot \underline{R}) |\underline{u}, \underline{R}\rangle + (\underline{u} \cdot \underline{R}) |\underline{y}, \underline{R}\rangle \dots\dots\dots (46b)$$

$$|d_3\rangle = (u^2 + y^2) |\underline{u}, \underline{y}\rangle$$

with

$$\begin{aligned} |\underline{a}, \underline{b}\rangle = \left\{ 3(\underline{a}_1 \underline{a}_2) (\underline{a}_3 \underline{a}_4) + 3(\underline{a}_1 \underline{a}_3) (\underline{a}_2 \underline{a}_4) - 2(\underline{a}_1 \underline{a}_4) (\underline{a}_2 \underline{a}_3) \right\} |s\rangle \\ \dots\dots\dots (47) \end{aligned}$$

It should be noted that ψ_1, ψ_2, ψ_3 are not orthogonal but it is simpler to deal with them in the above form than to attempt to construct orthogonal wave functions.

The s-state is completely symmetric under the interchange of the

space coordinates of all nucleons but the D-states are only symmetric under the interchange of all protons with all neutrons, i.e. in the notation of §3.

$$\begin{aligned} \text{s - state} \quad \alpha_p &= \alpha_N ; \beta_{pp} = \beta_{NN} = \beta_{pN} \\ &\dots\dots\dots (48a) \end{aligned}$$

$$\begin{aligned} \text{D - state} \quad \alpha_p &= \alpha_N ; \beta_{pp} = \beta_{NN} \neq \beta_{pN} \\ &\dots\dots\dots (48b) \end{aligned}$$

This type of wave function has been used in other calculations on the α -particle binding energy. Irving (1953) considered the s - state plus the principal D-state t_1 , whilst Clark (1954) used the s-state plus two D-states t_1 and t_2 .

In the following, the spin matrix elements have been evaluated by the method introduced by Irving (1953).

The wave functions are normalised as follows:-

$$\begin{aligned} \int t_s^2 d\tau &= \int t_1^2 d\tau = \int t_2^2 d\tau = \int t_3^2 d\tau = 1 \\ &\dots\dots\dots (49) \end{aligned}$$

$$\int t_a^2 d\tau = 1$$

Hence

$$\begin{aligned} N_s^{-2} &= \frac{\pi^4}{7.5.3.2} \int_0^\infty \rho^8 \phi_s^2(\rho) d\rho \quad ; \quad N_1^{-2} = \frac{\pi^4}{11.7.3.2} \times \\ &\quad \int_0^\infty \rho^{12} \phi_1^2(\rho) d\rho \dots\dots\dots (50) \end{aligned}$$

$$N_2^{-2} = \frac{\Pi^4}{13 \cdot 11 \cdot 5 \cdot 3 \cdot 2 \cdot 4} \int_0^\infty \rho^{16} \phi_2^2(\rho) d\rho; \quad N_3^{-2} = \frac{\Pi^4}{13 \cdot 11 \cdot 7 \cdot 3} \times$$

$$\int_0^\infty \rho^{16} \phi_3^2(\rho) d\rho$$

and

$$\int \psi_a^2 d\tau = a_s^2 + a_1^2 + a_2^2 + a_3^2 + 2a_1 a_2 \langle \psi_1 | \psi_2 \rangle + 2 a_2 a_3 \times$$

$$\langle \psi_2 | \psi_3 \rangle$$

$$+ 2 a_3 a_1 \langle \psi_3 | \psi_1 \rangle = 1 \dots\dots\dots (51)$$

where

$$\langle \psi_1 | \psi_2 \rangle = N_1 N_2 \frac{\Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2 \cdot 4} \int_0^\infty \rho^{14} \phi_1(\rho) \phi_2(\rho) d\rho$$

$$\langle \psi_2 | \psi_3 \rangle = N_2 N_3 \frac{\Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2 \cdot 4} \int_0^\infty \rho^{16} \phi_2(\rho) \phi_3(\rho) d\rho$$

$$\langle \psi_3 | \psi_1 \rangle = N_3 N_1 \frac{5 \cdot \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2} \int_0^\infty \rho^{14} \phi_3(\rho) \phi_1(\rho) d\rho$$

..... (52)

5. Bremsstrahlung - Weighted Cross Section.

For the α -particle $A = 2N = 2Z = 4$ and hence from (34)

$$G_b = \frac{4 \Pi^2}{3} \left(\frac{e^2}{\hbar c} \right) \langle R^2 \rangle_{\infty} \dots\dots\dots (53)$$

where

$$\langle R^2 \rangle_{\infty} = \int \psi_a^* R^2 \psi_a d\tau$$

$$= \sum_{i,j} a_i a_j \langle \psi_i | R^2 | \psi_j \rangle \quad i, j = s, 1, 2, 3. \quad \dots\dots\dots (54)$$

The various matrix elements are readily evaluated to give:-

$$\langle \psi_s | R^2 | \psi_s \rangle = \frac{N_s^2 \Pi^4}{9 \cdot 5 \cdot 7 \cdot 2^3} \int_0^\infty \rho^{10} \phi_s^2(\rho) d\rho$$

$$\langle \psi_s | R^2 | \psi_i \rangle = 0 = \langle \psi_i | R^2 | \psi_s \rangle \quad i = 1, 2, 3.$$

$$\langle \psi_1 | R^2 | \psi_1 \rangle = \frac{N_1^2 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 2^3} \int_0^\infty \rho^{14} \phi_1^2(\rho) d\rho$$

$$\langle \psi_2 | R^2 | \psi_2 \rangle = \frac{7 N_2^2 \Pi^4}{17 \cdot 15 \cdot 13 \cdot 11 \cdot 3 \cdot 2^6} \int_0^\infty \rho^{18} \phi_2^2(\rho) d\rho \quad \dots\dots\dots (55)$$

$$\langle \psi_3 | R^2 | \psi_3 \rangle = \frac{N_3^2 \Pi^4}{17 \cdot 13 \cdot 11 \cdot 7 \cdot 2^2} \int_0^\infty \rho^{18} \phi_3^2(\rho) d\rho$$

$$\langle \psi_1 | R^2 | \psi_2 \rangle = \frac{N_1 N_2 \Pi^4}{13 \cdot 11 \cdot 9 \cdot 7 \cdot 2^4} \int_0^\infty \rho^{16} \phi_1(\rho) \phi_2(\rho) d\rho$$

$$= \langle \psi_2 | R^2 | \psi_1 \rangle$$

$$\langle \psi_2 | R^2 | \psi_3 \rangle = \frac{5 N_2 N_3 \Pi^4}{17 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 2^4} \int_0^\infty \rho^{18} \phi_2(\rho) \phi_3(\rho) d\rho$$

$$= \langle \psi_3 | R^2 | \psi_2 \rangle$$

$$\begin{aligned} \langle t_3 | R^2 | t_1 \rangle &= \frac{N_3 N_1 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2^3} \int_0^\infty \rho^{16} \phi_3(\rho) \phi_1(\rho) d\rho \\ &= \langle t_1 | R^2 | t_3 \rangle \end{aligned}$$

6. Root Mean Square Radius. Matrix Elements.

Since, for the α -particle wave functions

$$N = Z, \quad \alpha_p = \alpha_N; \quad \beta_{NN} = \beta_{pp}$$

from (48), and hence

$$R_c^2 = R_m^2 = R^2 \quad \dots\dots\dots (56)$$

the r.m.s. radius is given by (31) as

$$\begin{aligned} R^2 &= 1/16 \int t_a^* \rho^2 t_a d\tau \\ &= 1/16 \sum_{ij} a_i a_j \langle t_i | \rho^2 | t_j \rangle \quad ij = s, 1, 2, 3 \\ &\dots\dots\dots (57) \end{aligned}$$

The various matrix elements are:-

$$\langle t_s | \rho^2 | t_s \rangle = \frac{N_s^2 \Pi^4}{7 \cdot 5 \cdot 3 \cdot 2^5} \int_0^\infty \rho^{10} \phi_s^2(\rho) d\rho$$

$$\langle t_s | \rho^2 | t_i \rangle = \langle t_i | \rho^2 | t_s \rangle = 0 \quad i = 1, 2, 3$$

$$\begin{aligned}
\langle t_1 | \rho^2 | t_1 \rangle &= \frac{N_1^2 \Pi^4}{11 \cdot 7 \cdot 3 \cdot 2^5} \int_0^\infty \rho^{14} \phi_1^2(\rho) d\rho \\
\langle t_2 | \rho^2 | t_2 \rangle &= \frac{N_2^2 \Pi^4}{13 \cdot 11 \cdot 5 \cdot 3^2 \cdot 2^8} \int_0^\infty \rho^{18} \phi_2^2(\rho) d\rho \\
\langle t_3 | \rho^2 | t_3 \rangle &= \frac{N_3^2 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2^4} \int_0^\infty \rho^{18} \phi_3^2(\rho) d\rho \\
\langle t_1 | \rho^2 | t_2 \rangle &= \frac{N_1 N_2 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2^6} \int_0^\infty \rho^{16} \phi_1(\rho) \phi_2(\rho) d\rho \\
&= \langle t_2 | \rho^2 | t_1 \rangle \dots\dots\dots (58) \\
\langle t_2 | \rho^2 | t_3 \rangle &= \frac{N_2 N_3 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2^6} \int_0^\infty \rho^{18} \phi_2(\rho) \phi_3(\rho) d\rho \\
&= \langle t_3 | \rho^2 | t_2 \rangle \\
\langle t_3 | \rho^2 | t_1 \rangle &= \frac{N_3 N_1 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2^5} \int_0^\infty \rho^{16} \phi_3(\rho) \phi_1(\rho) d\rho \\
&= \langle t_1 | \rho^2 | t_3 \rangle
\end{aligned}$$

Comparing 6_b with R^2 it is found that relation (43)

$$6_b = \frac{4\Pi^2}{3} \left(\frac{e^2}{\hbar c} \right) \frac{NZ}{A-1} R^2 \dots\dots\dots (43)$$

holds only for the s-state t_{s_1} , as was expected since only for this state is relation (40) satisfied i.e.

$$\beta_{pp} = \beta_{pN} = \beta_{NN}$$

However, as will be shown by explicit calculation in §9, the inclusion of the various D-states makes little or no difference to either 6_p or R^2 . Hence (43) can be used to compare the experimental values of 6_p and Rm^2 .

7. Integrated Cross Section.

The integrated cross section 6_{int} is given by

$$6_{int} = \frac{2\pi^2 e^2 \hbar}{mc} \left\{ 1 - \frac{4m}{\hbar^2} I \right\} \dots\dots\dots (59)$$

as is seen by using (26) with $A = 2N = 2Z = 4$.

Here I is given by

$$I = \int t_a^x U(13) Z_{13}^2 M_{13} t_a d\tau = \sum a_i a_j \langle t_i | I | t_j \rangle \dots\dots\dots (60)$$

The various matrix elements are:-

$$\langle t_s | I | t_s \rangle = \frac{N_s^2}{3} \frac{\pi^4}{2^5} (m + \frac{1}{2}\hbar) \int_0^\infty \phi_s^2(\rho) \rho^{10} d\rho \int_0^{\pi/2} \cos^4\theta \sin^5\theta \times_c V_c \left(\frac{\rho \cos\theta}{\sqrt{2}} \right) d\theta$$

Here only the central force gives a contribution. In the $\langle t_s | I | t_D \rangle$ terms only the tensor force contributes.

$$\begin{aligned} \langle t_s | I | t_1 \rangle &= (-1)^{m_1 N_1 N_s} \frac{\pi^4}{4^3} \int_0^\infty \phi_s(\rho) \phi_1(\rho) \rho^{12} d\rho \\ &\quad \int_0^{\pi/2} \cos^6\theta \sin^5\theta V_T \left(\frac{\rho \cos\theta}{\sqrt{2}} \right) d\theta \\ \langle t_s | I | t_2 \rangle &= m^1 N_2 N_s \frac{\pi^4}{32^9} \int_0^\infty \phi_s(\rho) \phi_2(\rho) \rho^{14} d\rho \times \end{aligned}$$

$$\begin{aligned}
& \times \int_0^{\pi/2} \cos^6 \theta \sin^5 \theta V_T \left(\frac{\rho \cos \theta}{\sqrt{2}} \right) \left\{ 5 \sin^2 \theta - 6 \cos^2 \theta \right\} d\theta \\
\langle \uparrow_s | I | \uparrow_3 \rangle &= (-1)^{m_1 N_3 N_s} \frac{\pi^4}{3 \cdot 2^9} \int_0^\infty \phi_s(\rho) \phi_3(\rho) \rho^{14} d\rho \\
& \times \int_0^{\pi/2} \cos^6 \theta \sin^5 \theta V_T \left(\frac{\rho \cos \theta}{\sqrt{2}} \right) \left\{ 13 \sin^2 \theta + 6 \cos^2 \theta \right\} d\theta \\
& \dots\dots\dots (61)
\end{aligned}$$

The terms $\langle \uparrow_i | I | \uparrow_j \rangle$ $i, j = 1, 2, 3$ must now be considered.

These terms only contribute a small fraction of the sum rule since the percentage D-state in the α -particle is small and also since the integrals occurring are not unduly large. Only the $\langle \uparrow_1 | I | \uparrow_1 \rangle$ term has been calculated since it is the most important. Both central and tensor forces contribute.

$$\begin{aligned}
\langle \uparrow_1 | I | \uparrow_1 \rangle_c &= N_1^2 (m+h) \frac{\pi^4}{2^8} \int_0^\infty \rho^{14} \phi_1^2(\rho) d\rho \int_0^{\pi/2} \cos^4 \theta \sin^5 \theta \times \\
& V_C \left(\frac{\rho \cos \theta}{\sqrt{2}} \right) d\theta \left\{ \frac{25}{16} \sin^4 \theta - \frac{5}{3} \sin^2 \theta \cos^2 \theta + \cos^4 \theta \right\} \\
\langle \uparrow_1 | I | \uparrow_1 \rangle_T &= m^1 N_1^2 \frac{\pi^4}{3 \cdot 2^8} \int_0^\infty \rho^{14} \phi_1^2(\rho) d\rho \int_0^{\pi/2} \sin^5 \theta \cos^6 \theta \times \\
& V_T \left(\frac{\rho \cos \theta}{\sqrt{2}} \right) d\theta \times \left\{ 7 \sin^2 \theta - 6 \cos^2 \theta \right\}
\end{aligned}$$

6_{int} , 6_b and R^2 will now be evaluated for the various wave functions introduced in §4.

8. Internucleon Potential and Wave Function Parameters.

The internucleon potential is taken to be of the form

$$V(r_{12}) = -V_0 \left[1 + \frac{1}{2} g \left(\frac{r_{12}}{r_0} \right)^{-1} \right] J_c \left(\frac{r_{12}}{r_c} \right) - \gamma V_0 S_{12} J_T \left(\frac{r_{12}}{r_T} \right)$$

..... (62)

with

$$J_c(x) = J_T(x) = \frac{e^{-x}}{x}$$

Extensive sets of parameters which satisfy the deuteron binding energy and quadrupole moment, have been tabulated by Feshbach and Schwinger (1951).

The sets used in the various binding energy calculations are shown in Table V.

TABLE V.

No	r_c 10^{-13} cm	r_T 10^{-13} cm	V_0 Mev	γ	g	% D-state in Deuteron	Effective Triplet Range	Authors
1	1.184	2.757	59.54	0.107	0.110	2.1	1.85	Abraham et al.
2	1.184	2.120	54.53	0.231	0.074	2.7	1.79	Abraham et al.
3	1.184	1.67	46.1	0.54	-0.004			Irving
4	1.184	1.58	42.7	0.69	-0.044			Clark
5	1.184	1.532	39.49	0.836	-0.088	3.9	1.71	Abraham et al.
6	1.184	1.379	30.65	1.507	-0.256	4.5	1.66	Abraham et al.

The details of the ~~six~~ wave functions used are as follows:-

(i) Irving.

$$t_{\alpha} = \frac{1}{(1+C^2)^{\frac{1}{2}}} \left\{ t_s + C t_1 \right\}$$

with

$$\phi_s(\rho) = \exp - \alpha \rho \quad ; \quad \phi_1(\rho) = \exp - \beta \rho \quad \dots\dots\dots (63)$$

$$C = -0.162 \quad 2\sqrt{2} \alpha r_c = 4.0 \quad 2\sqrt{2} \beta r_c = 6.0$$

This gave a binding energy of -24.3 Mev using potential 3.

(ii) Abraham Cohen & Roberts.

$$t_{\alpha} = \sum_i a_i t_i \quad i = s, 1, 2, 3. \quad \dots\dots\dots (64)$$

with

$$\phi_s(\rho) = \exp - \alpha \rho \quad ; \quad \phi_1(\rho) = \phi_2(\rho) = \phi_3(\rho) = \exp - \beta \rho$$

where the parameters a_i , α , β , are given in Table VI.

TABLE VI.

Potential	$x=2\sqrt{2}\alpha r_c$	$y=2\sqrt{2}\alpha r_c$	a_s	a_1	a_2	a_3	Binding Energy.
1	4.82	6.5	0.9891	-0.1245	0.1086	-0.0513	-46.6
2	4.48	6.5	0.9843	-0.1399	0.1198	-0.0821	-40.3
5	3.8	5.2	0.9710	-0.2325	0.1787	-0.0731	-23.5
6	3.0	5.2	0.9731	-0.1395	0.1797	-0.1289	-14.1

(iii) Clark.

$$\psi_a = N_0 \left\{ |s\rangle - N_1 \lambda \eta^{-2} |d_1\rangle + N_2 \mu \eta^{-4} |d_2\rangle \right\} \eta^{-4} \psi(\eta) \quad \dots\dots\dots (65)$$

where

$$N_0^2 = \frac{105}{321^4} ; N_1^2 = \frac{11}{80} ; N_2^2 = \frac{429}{224} ; \eta = \frac{\rho}{2}$$

λ and μ are functions of η found by minimising the energy and are given in Figure 1 of Clark's paper.

$$\langle \psi_a | \psi_a \rangle = \frac{2^3}{a} \int_0^\infty \phi^* \phi dx$$

where $x = a \eta$ $a = \sqrt{2}/r_c$ and $\phi = r^{\frac{1}{2}} \psi(\eta)$

$$\Gamma = 1 + \lambda^2 + \mu^2 - 4K \lambda \mu \quad K^2 = \frac{15}{182}$$

To a first approximation for ϕ , Clark finds

$$\phi_0 = (1 + \beta e^{-bt})^c \sinh^p t \cosh^{-q} t$$

where $t = ax$. The values of the parameters to give the best fit for the binding energy are:-

$$\begin{array}{lll} \alpha = 0.5 & p = 4.0 & b = 2.557 \\ \beta = 1.618 & q = 6.0 & c = 3.120 \end{array}$$

This wave function gives a value of -29.59 Mev for the binding energy using potential 4.

9. Numerical Results.

The following matrix elements are tabulated for the Exponential type (I.41) wave function in Table VII.

$$(i) \langle t_i | t_j \rangle \quad - \text{normalisation.}$$

$$(ii) \langle t_i | R_Z^2 | t_j \rangle \quad - \text{Bremstrahlung Weighted Cross Section.}$$

$$(iii) \langle t_i | \rho^2 | t_j \rangle \quad - \text{R.m.s. radius.}$$

For the Clark Wave function.

$$\langle t_a | t_a \rangle = 2^3 \int_0^\infty \phi^x \phi \, d\eta \quad ; \langle t_a | \rho^2 | t_a \rangle = 2 \int_0^\infty \eta^2 \phi^2 \, d\eta$$

$$\langle t_a | R_Z^2 | t_a \rangle = \int_0^\infty \eta^2 t^2 \cdot 2^3 \left[\frac{1}{9} + \frac{1}{13} \lambda^2 + \frac{7}{51} \mu^2 - \frac{4}{9} K \lambda \mu \right] d\eta$$

..... (66)

TABLE VII.

$\langle i j \rangle$	$\langle t_i t_j \rangle$	$\langle t_i R_z^2 t_j \rangle$	$\langle t_i p^2 t_j \rangle$
$\langle s s \rangle$	$\frac{N_s^2 \Pi^4}{7 \cdot 5 \cdot 3 \cdot 2} \frac{18}{(2\alpha)^9} = 1$	$\frac{5}{2} \frac{1}{(2\alpha)^2}$	$\frac{9 \cdot 5}{2^3} \frac{1}{(2\alpha)^2}$
$\langle s 1, 2, 3 \rangle$	0	0	0
$\langle 1 1 \rangle$	$\frac{N_1^2 \Pi^4}{11 \cdot 7 \cdot 3 \cdot 2} \frac{12}{(2\beta)^{13}} = 1$	$\frac{7}{2} \frac{1}{(2\beta)^2}$	$\frac{13 \cdot 7}{2^3} \frac{1}{(2\beta)^2}$
$\langle 2 2 \rangle$	$\frac{N_2^2 \Pi^4}{13 \cdot 11 \cdot 9 \cdot 5 \cdot 2} \frac{16}{(2\beta)^{17}} = 1$	$\frac{21}{2} \frac{1}{(2\beta)^2}$	$\frac{17 \cdot 9}{2^3} \frac{1}{(2\beta)^2}$
$\langle 3 3 \rangle$	$\frac{N_3^2 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3} \frac{16}{(2\beta)^{17}} = 1$	$\frac{9}{2} \frac{1}{(2\beta)^2}$	$\frac{17 \cdot 9}{2^3} \frac{1}{(2\beta)^2}$
$\langle 1 2 \rangle$	$\frac{N_1 N_2 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2} \frac{14}{(2\beta)^{15}} = \frac{1}{2}$	$\frac{10}{3} \frac{1}{(2\beta)^2}$	$\frac{15}{2} \frac{1}{(2\beta)^2}$
$\langle 2 3 \rangle$	$\frac{N_2 N_3 \Pi^4}{13 \cdot 11 \cdot 9 \cdot 7 \cdot 2} \frac{16}{(2\beta)^{17}} = \sqrt{105}/21$	$\frac{5\sqrt{105}}{14} \frac{1}{(2\beta)^2}$	$\frac{17 \cdot 9 \sqrt{105}}{56} \frac{1}{(2\beta)^2}$
$\langle 3 1 \rangle$	$\frac{N_3 N_1 \Pi^4}{13 \cdot 11 \cdot 7 \cdot 3 \cdot 2} \frac{14}{(2\beta)^{15}} = \sqrt{105}/12$	$\frac{\sqrt{105}}{3} \frac{1}{(2\beta)^2}$	$\frac{5\sqrt{105}}{4} \frac{1}{(2\beta)^2}$

The matrix elements for the integrated cross section using the Exponential type wave function are as follows:-

$$\begin{aligned}
 \langle S | I | S \rangle &= N_S^2 (m + \frac{1}{2}h) \frac{\pi^4}{3 \cdot 2^5} \frac{V_c L^9}{(\nu/\sqrt{2})^{11}} I(x; 3, 2, 10) \\
 \langle S | I | 1 \rangle &= -N_1 N_S m^1 \frac{\pi^4}{2^6} \frac{V_T L^{11}}{(\mu/\sqrt{2})^{13}} I(z; 5, 2, 12) \\
 \langle S | I | 2 \rangle &= N_2 N_S m^1 \frac{\pi^4}{3 \cdot 2^8} \frac{V_T L^{13}}{(\mu/\sqrt{2})^{15}} \\
 &\quad \times \left\{ \frac{5}{2} I(z; 5, 3, 14) - 3 I(z; 7, 2, 14) \right\} \\
 \langle S | I | 3 \rangle &= (-1) N_3 N_S m^1 \frac{\pi^4}{3 \cdot 2^9} \frac{V_T L^{13}}{(\mu/\sqrt{2})^{15}} \dots\dots\dots (67) \\
 &\quad \times \left\{ 6 I(z; 7, 2, 14) + 13 I(z; 5, 3, 14) \right\} \\
 \langle 1 | I | 1 \rangle_o &= N_1^2 (m + h) \frac{\pi^4}{2^8} \frac{V_c L^{13}}{(\nu/\sqrt{2})^{15}} \\
 &\quad \times \left\{ \frac{25}{16} I(y; 3, 4, 14) - \frac{5}{3} I(y; 5, 3, 14) + I(y; 7, 2, 14) \right\} \\
 \langle 1 | I | 1 \rangle_T &= N_1^2 m^1 \frac{\pi^4}{3 \cdot 2^8} \frac{V_T L^{13}}{(\mu/\sqrt{2})^{15}} \\
 &\quad \times \left\{ 7 I(w; 5, 3, 14) - 6 I(w; 7, 2, 14) \right\}
 \end{aligned}$$

where

$$\begin{aligned} v &= \frac{1}{r_t}, \quad \mu = \frac{1}{r_t}; \quad x = 2\sqrt{2}r_c \alpha; \quad y = 2\sqrt{2}r_c \beta \\ w &= 2\sqrt{2}r_t \beta; \quad z = \sqrt{2}(\alpha + \beta) r_t \end{aligned} \quad \dots\dots\dots (68)$$

and

$$I(\alpha; p, q, r) = \int_0^1 \frac{t^p (1-t^2)^q}{(\alpha+t)^r} dt. \quad \dots\dots\dots (69)$$

This integral is evaluated in Appendix C.

For the Clark wave function, only the $\langle S | I | S \rangle$ term has been evaluated.

$$\begin{aligned} \langle S | I | S \rangle &= 2.3.5.7 \quad V_c \left(m + \frac{1}{2}h\right) \left(\frac{1}{\sqrt{2}j}\right)^3 \int_0^1 x t^2(x) dx \\ &\quad \left\{ A_3(x) - 2A_5(x) + A_7(x) \right\} \quad \dots\dots\dots (70) \end{aligned}$$

where

$$\begin{aligned} A_n(x) &= \int_0^1 e^{-xt} t^n dt = \frac{t^n}{x^{n+1}} \left[1 - e^{-xt} e_n(x) \right] \\ e_n(x) &= \sum_{m=0}^n \frac{x^m}{t^m} \end{aligned} \quad \dots\dots\dots (71)$$

Using the parameters given in §8, in the above expressions leads to the following results.

TABLE VIII

Potential No	r_t	R^2 10^{-26} cm^2 S-state	R^2 10^{-26} cm^2 S+3D states	6_b mb S-state	6_b mb S+3D states	Wave Function
1	2.757	0.68	0.67	0.87	0.87	Exponential
2	2.120	0.79	0.77	1.01	1.00	Exponential
3	1.67	0.99	0.99	1.27	1.23	Exponential (Irving)
4	1.58	0.95	-	1.20	-	Clark
5	1.532	1.09	1.06	1.40	1.39	Exponential
6	1.379	1.75	1.66	2.24	2.21	Exponential

It is seen that the explicit inclusion of the various D- states does not alter either the root mean square radius or the bremsstrahlung - weighted cross section by more than $\sim 3\%$. Thus it is valid to use equation (43) to relate these quantities experimentally even although the wave function is not completely symmetric in all nucleons.

For the α -particle

$$6_b = 0.1281 R_m^2 \quad (\text{cm}^2 \cdot 10^{-26}) \quad \dots\dots\dots (72)$$

Experimentally

$$6_b = 2.40 \pm 0.15 \text{ mb} \quad R_m = 1.41 \cdot 10^{-13} \text{ cm} \quad \dots\dots\dots (73)$$

Using (72) and the experimental value for R_m gives

$$\sigma_p = 2.54 \text{ mb.} \quad \dots\dots\dots (74)$$

Thus, it is seen that, within the experimental error, the charge distribution measured by high energy electron scattering is the same charge distribution as gives rise to the electric dipole absorption at moderate energies.

However, wave functions which are chosen by a variational calculation to give the correct binding energy, give too concentrated a nucleus. They are consistent neither with the electron scattering experiments nor with the photodisintegration experiments.

The integrated cross section is given by

$$\begin{aligned} \sigma_{int} &= \frac{2 \pi^2 e^2 \hbar}{mc} \left\{ 1 - \frac{4m}{\hbar^2} I \right\} \\ &= 60 \left[1 + \frac{4m}{\hbar^2} \left\{ (m + \frac{1}{2}\hbar) a_s^2 \langle S | I | S \rangle + 2 m^1 a_s a_1 \langle S | I | 1 \rangle \right. \right. \\ &\quad \left. \left. + (m + \hbar) a_1^2 \langle 1 | I | 1 \rangle_c + m^1 a_1^2 \langle 1 | I | 1 \rangle_T \right\} \right] \\ &= 60 \left[1 + m (A+C) + \hbar (\frac{1}{2}A+C) + m^1 B \right] \text{ Mev. mb.} \\ &\quad \dots\dots\dots (75) \end{aligned}$$

where only the principal D-state is considered. The matrix elements have been evaluated for the wave functions introduced in §8. The results are shown in Table IX, for $\frac{4m}{\hbar^2} \langle I | \rangle$.

Also tabulated for the five Exponential type wave functions are the constants A, B, C which are related to the coefficients m, m^1, h as shown in (75).

TABLE IX

Potential No	r_t	$-\frac{4m}{h^2} \langle S I S \rangle$	$-\frac{4m}{h^2} \langle S I 1 \rangle$	$-\frac{4m}{h^2} \langle 1 I 1 \rangle_c$	$-\frac{4m}{h^2} \langle 1 I 1 \rangle_T$
1	2.757	0.8815	-0.3400	0.4358	0.0353
2	2.120	0.8043	-0.4404	0.3988	0.0499
3	1.67	0.6698	-0.5607	0.3388	0.0740
4	1.58	0.5989	-	-	-
5	1.532	0.5677	-0.6101	0.2891	0.0905
6	1.379	0.4060	-0.6248	0.2244	0.1055

It will be seen that the $\langle S | I | S \rangle$ term is the dominant one since its integral is of the same order of magnitude as the other terms and since the percentage of S-state present is greater than 96.

TABLE X

Potential No	A	C	B	A + C	$C + \frac{1}{2}A$
1	0.8678	0.0068	0.0847	0.8746	0.4407
2	0.7883	0.0079	0.1237	0.7962	0.4021
3	0.6527	0.0087	0.1789	0.6614	0.3351
5	0.5370	0.0156	0.2813	0.5526	0.2841
6	0.3978	0.0201	0.1776	0.4179	0.2190

The values of A, B, C which are consistent with a binding energy of 28 MeV have been found by interpolation from the values tabulated.

These values are

$$A + C = 0.62. \quad \frac{1}{2}A + C = 0.33. \quad B = 0.26.$$

.....(76)

Three standard central force exchange mixtures are:-

TABLE XI

	w	m	b	h	6 int	
Rosenfeld	-0.13	+0.93	+0.46	-0.26	104 99.9	$m_1 = m_1^1$ $h = 0$ $m+h = m_1^1$
Inglis	0	+0.8	0.2	0	102.24	$m = m_1^1$
Serber	0.5	0.5	0	0	86.4	$m = m_1^1$

The various values for the integrated cross section are given in Column 6, using (75), (76) and the approximation of Column 7 for the tensor exchange parameters.

It is seen that the results are not inconsistent with the experimental results of Gorbunov and Spiridonov if a Serber exchange force is assumed.

$$\begin{aligned} 6_{\text{int}} (\text{exp}) &= 88 \pm 7 \text{ Mev. mb.} \\ 6_{\text{int}} (\text{theor}) &= 86.4 \text{ Mev. mb.} \end{aligned} \quad \text{..... (77)}$$

It seems probable that this agreement is fortuitous as it has already been seen that the wave functions used here give a serious discrepancy between theory and experiment in the case of the r.m.s. radius.

Bransden et al (1956) in a calculation on the elastic scattering of neutrons by tritons and ^3He , find that a Serber exchange mixture gives good agreement with experiment. This would imply that the Serber exchange mixture is a good approximation in the four-body problem.

Assuming that the Serber force is a good approximation then it is seen that the constants A, B, C must be relatively insensitive to the form of the wave function and of the potential.

The results obtained here should be compared with those of Levinger and Rustgi in a paper which appeared after this work had been completed. These authors used a potential of the form

$$\begin{aligned} V(ij) &= (w+m M_{ij} + b B_{ij} + h M_{ij} B_{ij}) V_c(ij) \\ &+ (w^{11} + m^{11} M_{ij} + b^{11} B_{ij} + h^{11} M_{ij} B_{ij}) V_T(ij) S_{ij} \end{aligned} \quad \text{..... (78)}$$

Since

$$B_{ij} S_{ij} = S_{ij} B_{ij} = S_{ij}$$

the tensor force may be written

$$[(w^{11} + b^{11}) + (m^{11} + h^{11}) M_{ij}] \times V_T(ij) S_{ij}$$

and therefore [cf (18)]

$$(w^{11} + b^{11}) \equiv w^1, \quad m^{11} + h^{11} \equiv m^1$$

Levinger and Rustgi find, for the integrated cross section (LR 27)

$$6_{\text{int}} = 60 \left\{ 1 + 0.6710(m + \frac{1}{2}h) + 0.1815(m^{11} + \frac{1}{2}h^{11}) + 0.0240(m + \frac{1}{2}h) - 0.0001(m^{11} + \frac{1}{2}h^{11}) \right\} \dots\dots\dots (79)$$

They used potential No. 3 and Irvings' wave function (63). It will be seen that the parameters m , m^1 , h , h^1 occur in (79) in a different manner than found previously in (75). This is due to the fact that the relation (LR 19).

$$\int \dagger_0^x B M \dagger_0 d\tau = \frac{1}{2} \int \dagger_0^x M \dagger_0 d\tau \dots\dots\dots (80)$$

stated to be generally true for even-even nuclei, by Levinger and Rustgi, is valid only for the S-states. It does not apply for the D-states or for the tensor force.

Apart from one or two small numerical differences, the results of Levinger and Rustgi are in agreement with those calculated here.

It should be noted here that the results obtained for R^2 , 6_b , and 6_{int} using the Clark wave function are in close agreement with those using the Exponential wave functions of Abraham et al.

The Clark wave function gives a slightly larger binding energy for the same potential, and gives a correspondingly more concentrated nucleus. However it has been shown by explicit comparison of the Clark wave function with the Exponential wave function giving the same binding energy, that the two wave functions are very similar in form and magnitude and have the same asymptotic form. This is a confirmation that the Exponential wave function has a reasonable asymptotic form, since the method used in deriving the Clark wave function ensures that it has the correct asymptotic form.

Since these wave functions are so similar and since the Exponential form is more readily dealt with in calculations, it will be sufficient in future work to consider only Exponential wave functions.

10. Conclusions.

The results of this chapter may be summarised as follows. For all wavefunctions which were derived to fit the ${}^4\text{He}$ binding energy, it is found that they give a very concentrated nucleus with an r.m.s. radius about two thirds of that observed experimentally. Also, using these wavefunctions gives a very low value for the bremsstrahlung - weighted cross section.

It has been shown that the r.m.s. radius and 6_p are closely related to one another and that the experimental results are in good agreement with one another. Thus the discrepancy must occur in the choice of wave function. It is necessary to have a wave function which gives the same binding energy but with a larger rms radius.

As has been seen, the integrated cross section depends on the exchange force used, as well as on the wave function. As the wave functions considered are not completely reliable, it is unlikely that much information concerning the exchange nature of the force can be obtained at the present stage. However if the results are relatively insensitive to the nuclear wave function (as seems likely), it will be seen that the results are consistent with a Serber exchange mixture. This is in agreement with the work of Bransden et al (1956) who find that the Serber exchange force gives good agreement with experiment for the four-body problem.

Before proceeding to investigate this discrepancy between experimental and theoretical results, it will be of interest to approach the problem from a different angle, by the direct evaluation of the matrix elements for the (γp) reaction.

The (γp) reaction has been investigated by several authors. The (γp) reaction has been investigated by several authors and has been determined variationally to give the best fit binding energies using central forces, the calculated (γp) section exhibited a maximum at an energy much higher than observed.

CHAPTER III

Direct Evaluation of the Matrix Element M_{ab}

1. Introduction.

In the preceding chapter, the closure relation (II.6) was applied so as to eliminate the final state wave functions from the matrix elements. In this chapter, assumptions will be made as to the nature of the final state and the matrix elements will be explicitly evaluated.

This approach has been considered previously by Flowers and Mandl (1951) and by Gunn and Irving (1951). These authors were able to account for the qualitative features of the observed cross section without assuming the existence of an excited state of ${}^4\text{He}$. They described the ejected proton by a plane wave i.e. the Born approximation. Gunn and Irving used simple analytic wave functions for ${}^3\text{H}$ and ${}^4\text{He}$ of the Gaussian and Irving forms. They found that if the parameters μ_α and μ_T were determined variationally to give the best ${}^3\text{H}$ and ${}^4\text{He}$ binding energies using central forces, the calculated (γ_p) cross section exhibited a maximum at an energy much higher than that observed.

In the following sections the effect of the final state interaction is considered.

Gunn and Irving have suggested that better agreement with experiment would be obtained by using wave functions derived in a

variational calculation with a mixture of central and tensor forces.

This is also considered in the following sections.

2. Photodisintegration Cross Section.

The probability of emission of an electric multipole quantum $\hbar \omega$ of order l, m is from (I.15)

$$T_E(l, m) = \frac{8 \pi (l+1)}{1 [(2l+1)!!]^2} \frac{K^{2l+1}}{\hbar} |Q_{lm}(ab)|^2 \dots \dots \dots (1)$$

where $K = \frac{\omega}{c}$ the wave number of the photon.

The total cross section for the radiative capture of a proton by a nucleus is

$$\sigma_{CAP} = \frac{1}{v} T(l, m) \dots \dots \dots (2)$$

where v is the relative velocity

$$v = \frac{\hbar k}{M^1} \dots \dots \dots (3)$$

with $\hbar k$ the momentum and M^1 the reduced mass of the proton.

The cross section for the photodisintegration is then found by applying the detailed balance theorem for inverse reactions. This gives for the total cross section

$$\begin{aligned} \sigma_{DIS}(l, m) &= \frac{k^2}{K^2} \sigma_{CAP}(l, m) \\ &= \frac{k^2}{K^2} \frac{1}{v} T(l, m) \dots \dots \dots (4) \end{aligned}$$

Hence the electric dipole cross section is

$$\sigma_{\text{dis}} = \frac{4}{3} \left(\frac{M^1 e^2}{\hbar^2} \right) kK \left| \sum_{i=1}^Z \int \phi_b^* \mathbf{r}_i \phi_a d\tau \right|^2 \dots\dots\dots (5)$$

If the initial photon beam is unpolarised, the cross section must be multiplied by the statistical weight ($\frac{1}{2}$) before comparison with experiment.

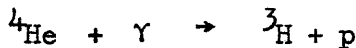
Hence

$$\sigma_{\text{DIS}} = \frac{2}{3} \left(\frac{M^1 e^2}{\hbar^2} \right) kK \left| M_{ab} \right|^2 \dots\dots\dots (6)$$

where

$$M_{ab} = \sum_{i=1}^Z \int \phi_b^* \mathbf{r}_i \phi_a d\tau \dots\dots\dots (7)$$

For the reaction



the E1 cross section is given by (6), (7) with $Z = 2$ and the reduced mass $M^1 = \frac{3}{4}m$ where m is the nucleon mass.

By the conservation of energy

$$E_\gamma + E_\alpha = E_p + E_t$$

where E_α, E_t are the binding energies of the α -particle and triton respectively, and E_p is the energy of the proton in the centre of mass system.

$$E_p = \frac{\hbar^2}{2M^1} k^2 \quad ; \quad E_\gamma = \hbar c K$$

Therefore

$$K = \frac{1}{\hbar c} \frac{\hbar^2}{2M^1} [k^2 + \lambda^2] \quad ; \quad \lambda^2 = \frac{2M^1}{\hbar^2} [E_\alpha - E_t] \dots\dots\dots (8)$$

Hence finally

$$6_{\text{DIS}} = \frac{1}{3} \left(\frac{e^2}{\hbar c} \right) k (k^2 + \lambda^2) |M_{ab}|^2 \dots\dots\dots (9)$$

Assuming for the moment central forces, the ground state of the α -particle is a 1S_0 state. Only the spatially symmetric part of the wave function will be considered since it is known that the ground states of the three and four particle nuclei consist largely of antisymmetric spin states.

The initial α -particle wave function may then be written (using the convention that 1,2 denote protons, 3,4 neutrons)

$$\phi_a = \psi_a = \phi_a (\overline{12}, \overline{34}) \phi_0 (\tilde{12}, \tilde{34}) \dots\dots\dots (10)$$

with

$$\int \left\{ \phi_a (\overline{12}, \overline{34}) \right\}^2 d\tau = 1$$

The bar denotes symmetry with respect to the interchange of the pair of particles. The singlet spin function $\phi_0 (\tilde{12}, \tilde{34})$ is

$$\begin{aligned} \phi_0 (\tilde{12}, \tilde{34}) = \frac{1}{2} \left\{ \alpha(1) \beta(2) - \beta(1) \alpha(2) \right\} \\ \left\{ \alpha(3) \beta(4) - \beta(3) \alpha(4) \right\} \dots\dots\dots (11) \end{aligned}$$

where α, β are the usual spin eigenfunctions.

The final state, proton plus triton, may be represented by a resonating group wave function (Wheeler 1937).

$$\phi_b = \frac{1}{\sqrt{2}} (1-P_{12}) \phi_T (2, \overline{34}) F(1) \phi_o (\tilde{12}, \tilde{34}) \dots\dots\dots (12)$$

where P_{12} is the Heisenberg exchange operator and $F(1)$ describes the motion of the proton relative to the triton.

ϕ_a and ϕ_T are the symmetric spatial wave functions for ${}^4\text{He}$ and ${}^3\text{H}$ respectively.

Hence

$$\begin{aligned} M_{ab} &= \frac{1}{\sqrt{2}} \int \left\{ (1-P_{12}) \phi_T (2, \overline{34}) F(1) \phi_o (1234) \right\} (\underline{z}_1 + \underline{z}_2) \\ &\quad \times \left\{ \phi_a (\overline{12}, \overline{34}) \phi_o (\tilde{12} \tilde{34}) \right\} d\tau \\ &= \sqrt{2} \int \phi_T (2, \overline{34}) F(1) (\underline{z}_1 + \underline{z}_2) \phi_a (\overline{12} \overline{34}) d\tau_{1234} \dots\dots\dots (13) \end{aligned}$$

where $\int d\tau_{1234}$ is the integral over space coordinates only.

Introducing the coordinate system (6) of Appendix A

$$\underline{u} = \underline{r}_4 - \underline{r}_3, \quad \underline{X} = \underline{r}_2 - \frac{1}{2} (\underline{r}_3 + \underline{r}_4), \quad \underline{r} = \underline{r}_1 - \frac{1}{3} (\underline{r}_1 + \underline{r}_2 + \underline{r}_3) \dots\dots\dots (14)$$

gives

$$M_{ab} = \frac{1}{\sqrt{2}} \int \phi_T (\underline{X}, \underline{u}) \phi_a (\underline{r}, \underline{X}, \underline{u}) F(\underline{r}) \left\{ \underline{r}_B + \frac{4}{3} \underline{x}_B \right\} d\underline{r} d\underline{X} d\underline{u} \dots\dots\dots (15)$$

It should be noted that $F(1)$ is now a function of \underline{r} only. In the Born approximation, the ejected proton is represented by a plane wave

$$F(1) = F(\underline{r}) = \exp i \underline{k} \cdot \underline{r} \quad \dots\dots\dots (16)$$

As ϕ_α and ϕ_T represent S-states, the proton is ejected into a P-state so that

$$F(\underline{r}) = \frac{1}{kr} f_1(r) P_1(\cos \theta) \quad \dots\dots\dots (17)$$

$$\text{and } f_1(r) \sim \sin(kr - \frac{1}{2}\pi - \alpha \log(2kr) + \delta) \quad \dots\dots\dots (18)$$

and

$$\alpha = \frac{3me^2}{4\hbar^2 k}$$

3. Interaction in the Final State.

In a calculation on the elastic scattering of neutrons by tritons, Bransden Robertson and Swan (1956) have used the resonating group method to derive integro-differential equations, for which accurate numerical solutions were obtained with the aid of an electronic computer. This work has recently been extended by Bransden and Robertson (1958) to proton triton scattering.

For the $(p + t)$ state these authors used

$$\phi_p(12 \ 34) = \frac{1}{\sqrt{2}} (1-P_{12}) \phi_T(2, \overline{34}) F(1) \phi_o(1234)$$

as (12) above.

$$\dots\dots\dots (19)$$

$F(1)$ was then expanded as a sum of partial waves

$$F(r) = \sum_n r^{-1} f_n(r) P_n(\cos \theta) \dots\dots\dots (20)$$

The $f_n(r)$ were obtained numerically subject to the boundary conditions.

Sinceⁱⁿ the photodisintegration, the proton is ejected in a P-state, only the function $f_1(r)$ is required in calculating the matrix element.

$f_1(r)$ has been found over a range of proton energies up to 40 Mev.

Bransden et al used a two body interaction of the form

$$V(ij) = (w + m M_{ij} + b B_{ij} + h M_{ij} B_{ij}) V(ij) + t_{ij} V_c(ij) \dots\dots\dots (21)$$

where V_c is the Coulomb interaction and $t_{ij} = 1$ if both i, j denote protons, but is zero otherwise. w, m, b, h are constants such that

$$m + w + b + h = 1 \quad m + w - b - h = x$$

where x is the ratio of singlet and triplet interactions between neutron and proton in an even state.

Two particular cases of this potential were investigated namely

(i) Symmetrical exchange force

$$\begin{aligned} m &= 2b = \frac{1}{3} (1 + 3x) \\ h &= 2w = \frac{1}{3} (1 - 3x) \end{aligned} \dots\dots\dots (22a)$$

(ii) Serber exchange force

$$\begin{aligned} m &= w = \frac{1}{4} (1 + x) \\ h &= b = \frac{1}{4} (1 - x) \end{aligned} \dots\dots\dots (22b)$$

The triplet even two body interaction was chosen of the Gaussian form with parameters consistent with an α -particle binding energy of 27 Mev and also consistent with the deuteron binding energy.

$$V(r) = V_0 \exp - \lambda r^2 \quad \dots\dots\dots (23)$$

$$V_0 = -45 \text{ Mev} \quad \lambda = 0.267 \cdot 10^{26} \text{ cm}^{-2} \quad x = 0.6$$

The triton is represented by a Gaussian wave function with $\mu_T^2 = 0.0718 \cdot 10^{26} \text{ cm}^{-2}$.

Comparison of the calculated $n - {}^3\text{H}$ and $n - {}^3\text{He}$ scattering cross sections with experiment suggests that (23) is a reasonable equivalent central potential for the four body system, when a Serber exchange factor is used.

For consistency, the initial ${}^4\text{He}$ ground state wave function was also taken to be of the Gaussian form (I.41) and the constant μ_α chosen so that the maximum of the photodisintegration cross section for a plane wave final state occurs at ~ 7 Mev above threshold to agree with experiment. This gives $\mu_\alpha^2 = 0.021 \cdot 10^{26} \text{ cm}^{-2}$ a value which leads to a very large root-mean-square radius $R = 2.5 \cdot 10^{-13} \text{ cm}$ for the α -particle.

Using these parameters, the (γp) cross section was evaluated in the following three cases.

- (i) Born approximation. Plane wave final state.
- (ii) Using the numerical wave function $f_1(r)$ determined with a Serber two body interaction.
- (iii) Using the numerical wave function $f_1(r)$ determined with a symmetrical two body interaction.

These cross sections are compared in Figure II.

It is seen that the interaction between proton and triton in the final state alters the detailed shape of the (γp) cross section but that the position of the maximum is not altered significantly. For the Serber potential, which is consistent with the $n - t$ scattering data, the maximum is quite close to that given by the plane wave final state, whilst the use of the symmetrical interaction gives a maximum at an energy about 2 Mev higher.

If μ_α is altered to the value that gives the best ${}^4\text{He}$ binding energy for the potential (23),

$$\mu_\alpha^2 = 0.0789 \cdot 10^{26} \text{ cm}^{-2} \quad \epsilon_\alpha = 26.7 \text{ Mev.}$$

then the position of the (γp) maximum cross section is much higher (~ 30 Mev above threshold).

Gunn and Irving found that in the Born approximation, the (γp) cross section exhibited a maximum at an energy 20 - 30 Mev above threshold, much higher than that observed ~ 7 Mev. To obtain the maximum at the correct energy, the size of the ${}^4\text{He}$ nucleus had to be increased considerably. This is in agreement with the very large value of the r.m.s radius required in the above calculation to give the maximum at 7 Mev.

A resonating group ground state wave function of the form (12) for ${}^4\text{He}$ has also been investigated. This wave function gives a binding energy as good as the Gaussian function ($\epsilon_\alpha = 26$ Mev) but has a much

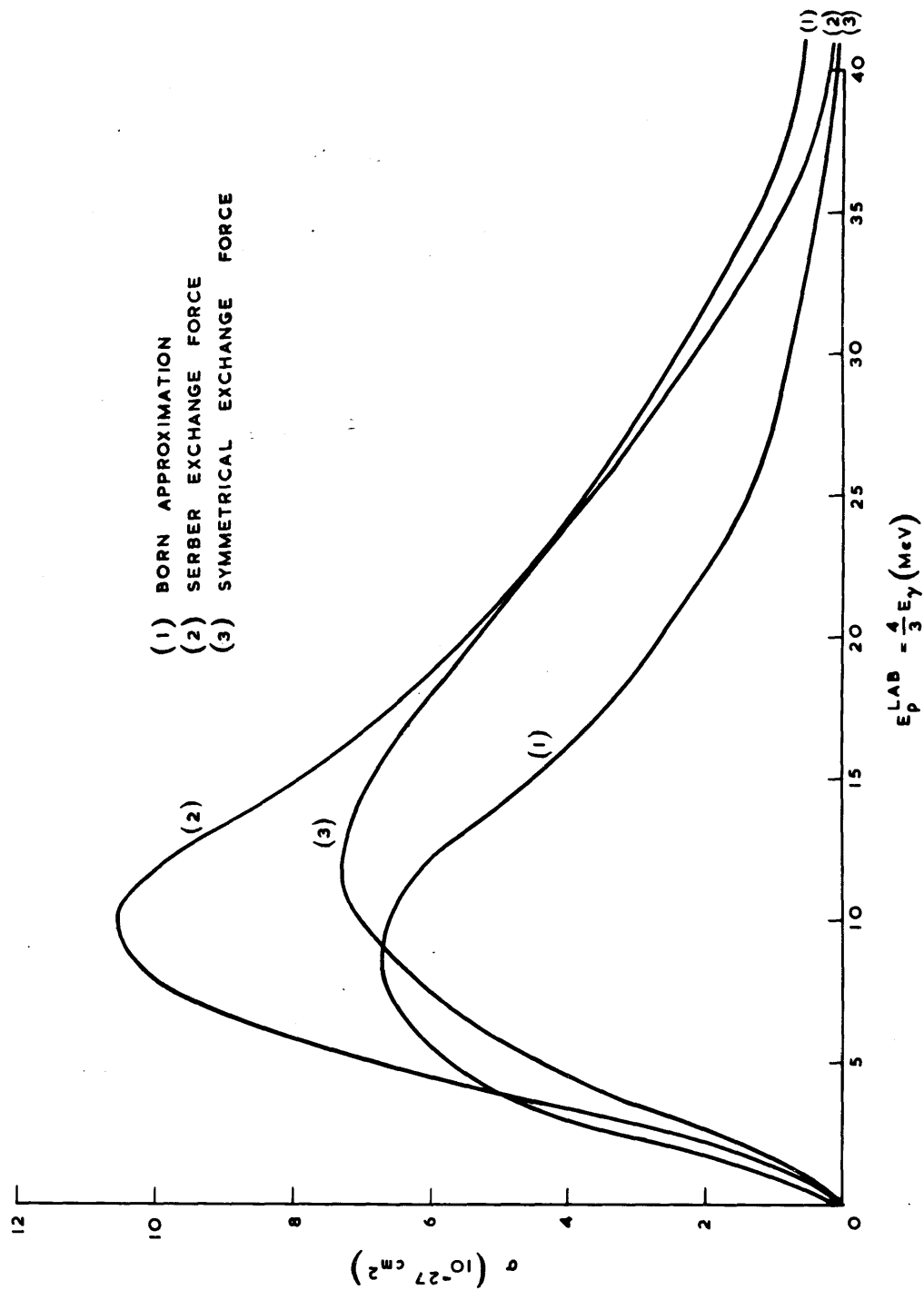


FIGURE 2. EFFECT OF FINAL STATE INTERACTION ON $\sigma(\gamma p)^3\text{H}$ CROSS-SECTION

better asymptotic form. However, in this case, the maximum cross section occurs at an energy which is still too high (~ 20 Mev above threshold) for both plane wave and interacting final states.

Since the introduction of the effect of the final state interaction alters the position of the maximum only slightly, it must be concluded that this interaction is not responsible for the discrepancy between the calculated and experimental (γp) cross section. This is, in effect a discrepancy between wave functions for the α -particle that are consistent with the binding energy and those that are consistent with the (γp) cross section.

In investigating this consistency problem, it should be sufficient to employ plane wave final states and this has been done in the following sections.

4. The Tensor Force.

It has been suggested by Gunn and Irving that the discrepancy might be removed if ^4He wave functions consistent with a two body interaction containing the tensor force, were used in calculating the cross section.

As discussed in II§4., Abraham et al. (1955) have investigated the binding energies of ^3H and ^4He with two body potentials containing central and tensor components, with Yukawa radial shapes. The S- and D-states were represented by the Exponential wave function.

In the Born approximation, the Exponential wave function gives for the (γp) cross section:-

$$6 (\gamma_P) = \frac{1}{3} \left(\frac{e^2}{\hbar c} \right) k^3 [k^2 + \lambda^2] \frac{\mu_a^{11} \mu_T^8}{\left\{ \frac{k^2}{12} + \frac{\mu_a^2}{4} + \frac{3\mu_T^2}{16} \right\}^{13}} \\ \times \frac{11^2 \cdot 9^2 \cdot 7^2 \cdot 5\sqrt{3}}{2^{25}} \left\{ f(z) \right\}^2 \\ \dots\dots\dots (24)$$

where

$$f(z) = \frac{1}{24} \frac{1}{\sqrt{z-1}} (585 z^2 - 975 z + 422) \\ - \frac{1}{11 \cdot 9 \cdot 7} \frac{1}{z^3 \sqrt{z-1}} (594 z^2 + 66 z + 8) \\ - \frac{5}{8} (39 z^2 - 52 z + 16) \sin^{-1} \frac{1}{\sqrt{z}}$$

$$\lambda^2 = - \frac{3m}{2\hbar^2} (E_a - E_T)$$

and

$$z = \frac{\mu_a^2 + \frac{k^2}{3}}{\mu_a^2 + \frac{k^2}{3} - \frac{3\mu_T^2}{4}}$$

Using the potential (5) of Table V, Abraham et al obtained a binding energy of 16.6 Mev using a ground state wave function composed of an S-state plus a 3.4% admixture of the principal D-state. This energy was increased to 23.5 Mev by the inclusion of five other D-states.

The variationally determined parameter for the S-state was found to be

$$\mu_a = 1.134 \cdot 10^{13} \text{ cm}^{-1}$$

The (γp) cross section calculated from (24) with $\mu_\alpha = \mu_T = 1.134 \times 10^{13} \text{ cm}^{-1}$ is shown in Figure III.

For comparison, the cross section for $\mu_\alpha = \mu_T = 0.8 \times 10^{13} \text{ cm}^{-1}$ and $\mu_\alpha = \mu_T = 0.5 \cdot 10^{13} \text{ cm}^{-1}$ were calculated and are shown also in Figure III. It is seen that the cross section for $\mu_\alpha = 1.134 \cdot 10^{13} \text{ cm}^{-1}$ rises much too slowly. For $\mu_\alpha = 0.5 \cdot 10^{13} \text{ cm}^{-1}$ the calculated cross section attains its maximum at approximately the experimental value but this value μ_α is far outside any value consistent with the ${}^4\text{He}$ binding energy.

If the tensor range is increased slightly, the calculated binding energy can be raised to 28 Mev but μ_α is then larger, $\mu_\alpha \sim 1.19 \times 10^{13} \text{ cm}^{-1}$ and the position of the maximum moves to an even greater energy.

5. Effect of the D-states.

In the preceding section, the effect of the tensor force was included implicitly in the value of the parameter μ_α occurring in the S-state wave function. In this section, the effect of the tensor force is considered explicitly by the inclusion of the D-states.

The α -particle wave function is of the form

$$\phi_\alpha = \phi_\alpha^S + \phi_\alpha^D \quad \dots\dots\dots (25)$$

where S, D, denote the S- and D- state parts of the wave function.

Similarly the final ($p + t$) state may be written

$$\phi_t = \phi_t^S + \phi_t^D \quad \dots\dots\dots (26)$$

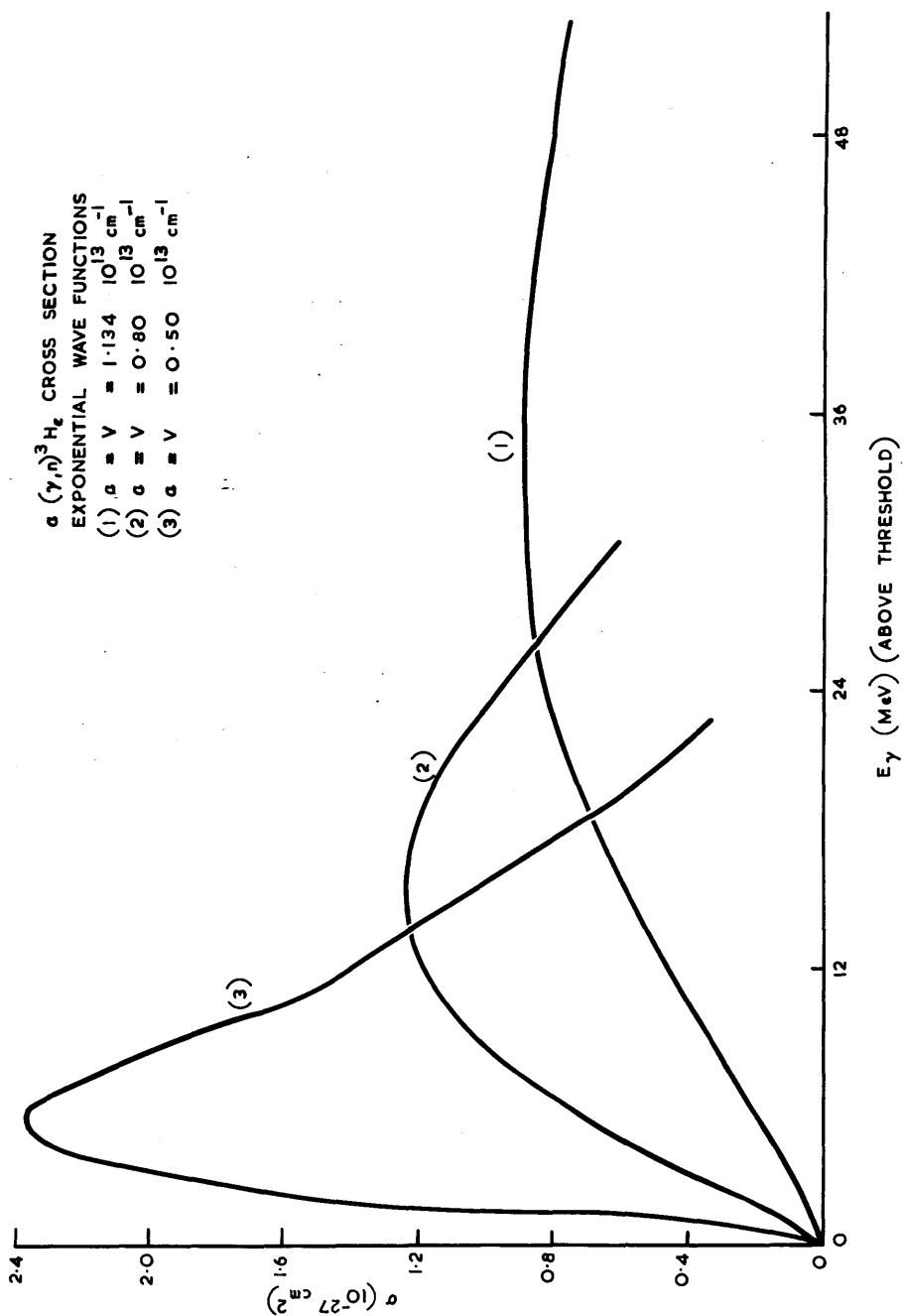


FIGURE 3

The percentage of P-states present will be negligible and so their effect may be omitted.

The matrix element is then given by

$$\begin{aligned}
 M_{ab} &= \sum_{i=1}^2 \int \phi_a^x z_i \phi_b d\tau \\
 M_{ab} &= \sum_{i=1}^2 \left\{ \int \phi_a^s z_i \phi_b^s d\tau + \int \phi_a^s z_i \phi_b^D d\tau \right. \\
 &\quad \left. + \int \phi_a^D z_i \phi_b^s d\tau + \int \phi_a^D z_i \phi_b^D d\tau \right\} \\
 &= \sum_{i=1}^2 \left\{ \langle s a | z_i | s b \rangle + \langle s a | z_i | D b \rangle \right. \\
 &\quad \left. + \langle D a | z_i | s b \rangle + \langle D a | z_i | D b \rangle \right\} \dots\dots\dots (27)
 \end{aligned}$$

The term $\langle s a | z_i | s b \rangle$ has been dealt with in the previous section. The last three terms will be considered here.

The D-states of the α -particle are all 5D_0 states, i.e. quintet spin states. The final state is (p + t). Considering the ground state of the triton to be an S-state i.e. $^2S_{1/2}$, the final state has either $S = 1$ i.e. a triplet spin state or $S = 0$ i.e. a singlet spin state. In either case, the final spin wave function is orthogonal to the initial spin wave function. Hence, as the electric dipole transition operator is independent of the spins, there is no contribution from the $\langle D a | z_i | s b \rangle$ term in (27).

Similarly if the α -particle is in an S- state and the triton is in a D-state, the spin wave functions are again orthogonal and so there is no contribution from the $\langle S \alpha | Z_i | D f \rangle$ term in (27).

The $\langle D \alpha | Z_i | D f \rangle$ term is non-zero but, since the D-state of ${}^4\text{He}$ is at most 4% of the total ground state and the D-state of ${}^3\text{H}$ at most 6%, the contribution will be small and may be neglected since the $\langle S \alpha | Z_i | S f \rangle$ terms are not abnormally small.

Hence it is seen from the results of this section and of the preceding one, that the effect of the tensor force in the photodisintegration of ${}^4\text{He}$ need only be taken into account implicitly through the value of the parameter μ_α . It need not be considered explicitly by the introduction of the D-state wave functions.

In contrast, the introduction of the tensor force is known to affect the binding energy appreciably and the D-state wave functions must be introduced explicitly. This is due to the fact that the binding energy is the difference of two large but approximately equal terms, the kinetic and the potential energies.

6. Conclusion.

Gunn and Irving have calculated the (γp) cross section in the Born approximation using Gaussian (I.41.i) and Irving (I.41.iii) type wave functions with various sets of parameters μ_α , μ_T .

To summarise these results and those of §4, with the Exponential wave functions, the variation of the position of the maximum of the cross section with the mean square radius R^2 , of the α -particle

is shown in Figure IV. For these three types of wave function the r.m.s. radii R , are given by

$$\begin{aligned}
 R_{\text{GAUSS}} &= \frac{3}{8} \frac{1}{\mu_{\alpha}} & R_{\text{EXP}} &= \frac{3}{4} \sqrt{\frac{5}{2}} \cdot \frac{1}{\mu_{\alpha}} \\
 R_{\text{IRV}} &= \sqrt{\frac{7}{8}} \frac{1}{\mu_{\alpha}} & & \dots\dots\dots (28)
 \end{aligned}$$

where only the S-states have been considered. It has been shown previously (II§9) that the addition of D-states does not alter R appreciably.

In all these calculations, the approximation $\mu_{\alpha} = \mu_T$ has been made but the (γp) cross section is relatively insensitive to the value of μ_T . In the case of the Gaussian wave function, μ_T occurs in the cross section only in a normalisation factor and therefore a change in μ_T leaves the position of the maximum unaltered.

It is clear from the figure that, for wave functions which are functions of ρ , the position of the maximum depends sensitively on the r.m.s radius R but is not very dependant on the shape of the wave function. Wave functions which are chosen to fit the binding energy give too small a value of R , and a very large value for the position of the maximum of the cross section.

7. Photodisintegration of the Lightest Nuclei.

At this point it is of interest to compare the results on the photodisintegration of the three and four body nuclei with that of the deuteron. It was noted in the introduction (I§3.) that the low energy

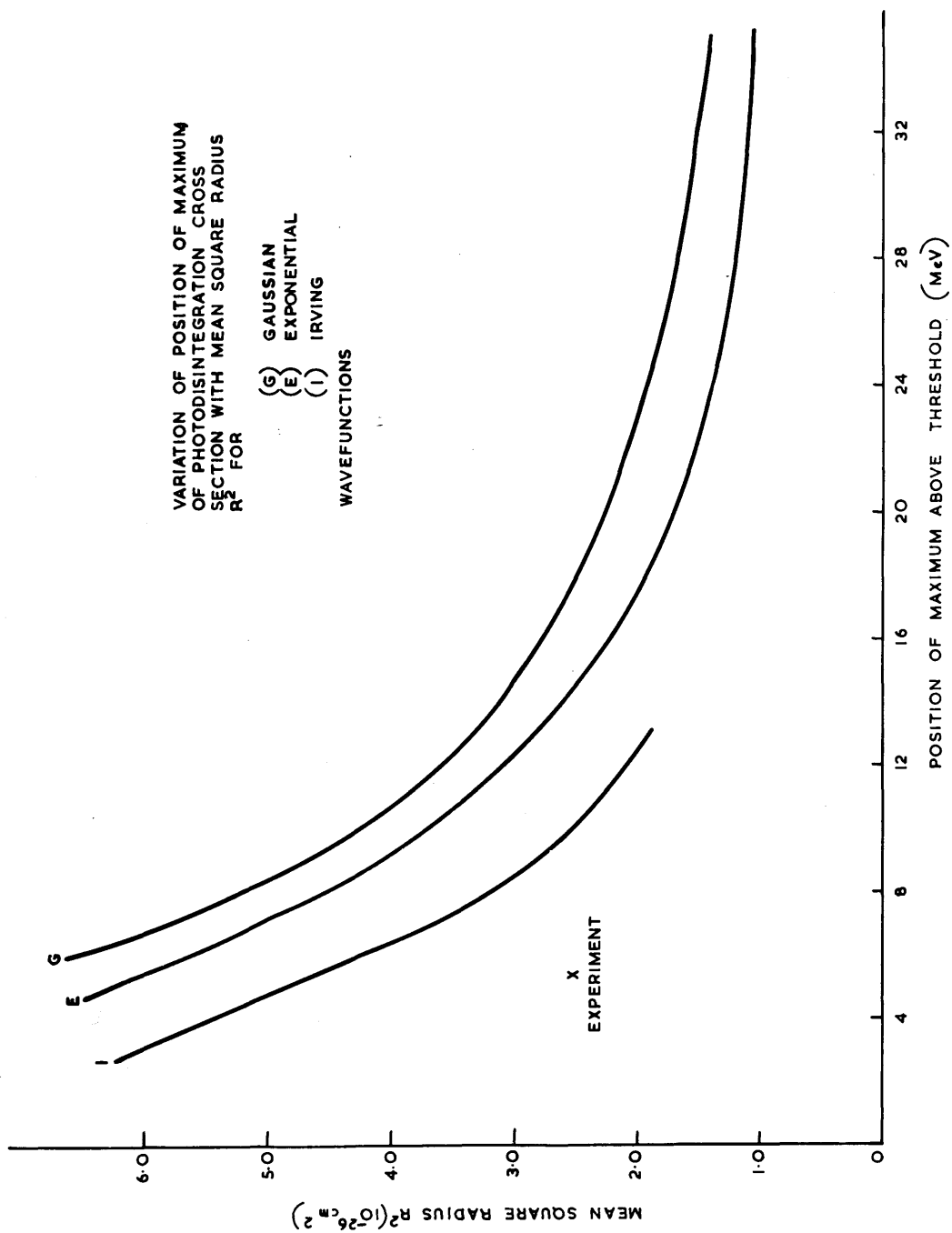


FIGURE 4

photodisintegration of the deuteron could be explained by the effective range theory. The nuclear potential can be described by two parameters, the well depth and range whilst the deuteron wave function can effectively be replaced by its asymptotic form.

This theory explains the position of the maximum of the cross section at 4.46 Mev above threshold, and the cross section up to about 25 Mev. However, the deuteron is a very weakly bound nucleus and has a correspondingly large r.m.s radius $R_d = 1.96 \cdot 10^{-13}$ cm. The two nucleons spend most of their time outside the range of the nuclear force and so it is to be expected that the low energy data is independent of the detailed form of the nuclear potential i.e. that the effective range theory is sufficient to describe the data.

In the case of the three and four body nuclei, the nucleons are much more tightly bound and the r.m.s radii are correspondingly smaller than for the deuteron. Thus it is to be expected that the properties of these nuclei will be more sensitive to the detailed shape of the potential. It has been seen that even when a wave function with a good asymptotic form is chosen for the α -particle, the photodisintegration cross section cannot be explained. When a wave function is chosen with a good asymptotic form e.g. the Clark wave function or the resonating group wave function of the form (12), the position of the maximum cross section occurs at 30 - 40 Mev above threshold.

These wave functions have been derived to give the correct α -particle binding energy using a potential consistent with the two body data.

Thus it is seen that the effective range theory is not sufficient to describe the three and four body data.

Possible reasons for the discrepancies are discussed in the following Chapter.

CHAPTER IV

Discussion of Results.

When considering the sum rules in Chapter II, it was found that the wave functions used gave too concentrated an α -particle and consequently too small a value for the bremsstrahlung weighted cross section. This is explained by the results on the direct evaluation of the matrix elements in Chapter III. It is seen that, for a wave function which gives the correct binding energy the position of the maximum occurs at ~ 30 Mev above threshold and that the cross section rises very slowly to the maximum.

Since the (γp) cross section has been shown experimentally to be the same as the (γn) cross section (neglecting the slight difference in the threshold region) and since these two reactions make up approximately ninety per cent of the total photon absorption, it is reasonable to assume that the cross section for photon absorption is of the same form as the (γp) cross section. As the bremsstrahlung-weighted cross section is given by

$$\sigma_b = \int \frac{\sigma(E)}{E} dE$$

it is seen that σ_b will be smaller for the theoretical cross section than for the experimental one which rises rapidly to a maximum at 7 Mev above threshold.

Using both approaches, sum rules and evaluation of the matrix elements, it has been found that, to obtain agreement between theory

and experiment it is necessary to increase the size of the α -particle as given by the theoretical wave functions. This will have the effect of giving agreement for the bremsstrahlung-weighted cross section since it is closely related to the r.m.s radius and since the experimental results are consistent. It is also expected that this increase will reduce the value of the position of the maximum of the (γp) cross section towards the experimental result.

Thus to resolve the discrepancy, a wave function must be found for the α -particle which gives the same binding energy as those used previously but has a root mean square radius about 50% greater.

It has been shown that the position of the maximum is not very sensitive to the type of wave-function used but that the maximum occurs earlier for the Irving type wave function than for the Gaussian and Exponential types with the same r.m.s radius.

For all wave functions which are functions of ρ , it has been found that to obtain the correct binding energy, a very concentrated nucleus must be used. This discrepancy may be due to the fact that the assumption

$$t_{\alpha} = t_{\alpha}(\rho)$$

is poor. However, a similar discrepancy has arisen in the consideration of the three-body nuclei.

If two body central forces are used with depths and ranges adjusted to fit the two body scattering data, the theoretical binding energy of ${}^3\text{H}$ is too large, as also is the Coulomb energy of ${}^3\text{He}$.

Pease and Feshbach (1951, 1952) have taken the tensor force into account and using a reasonable range for the tensor force so as to give the experimental binding energy of ${}^3\text{H}$, they find that the Coulomb energy still remains too large ($\sim 33\%$).

As the Coulomb energy is given by

$$E_{\text{COUL}} = e^2 \int \psi^* \frac{1}{|r_{12}|} \psi d\tau$$

it is seen that by increasing the size of the nucleus, the Coulomb energy would be reduced.

Thus it is seen that, for both the three and four particle nuclei, wave functions derived to fit the binding energy by means of the consistency problem, give too concentrated nuclei. As this is true in the three body case for several different forms of trial wave function, and not only for those which are functions of ρ , it would seem that the discrepancy is due not to the form of the wave function but to the form of the internucleon potential.

The potential used in the previous calculations was of the form

$$V(r_{12}) = -V_0 \left[1 + \frac{1}{2}g \begin{pmatrix} 6_1 & 6_2 & -1 \end{pmatrix} \right] J_c \left(\frac{r}{r_0} \right) - \gamma V_0 S_{12} J_T \left(\frac{r}{r_T} \right)$$

with

$$J_c(x) = J_T(x) = \frac{e^{-x}}{x}$$

This form of potential (with appropriate parameters) is consistent with all the low energy data. However this interaction fails when high energy data ($E > 50$ Mev) are considered. There are two main approaches to the problem of finding an internucleon potential to fit the experimental data at high energies (up to ~ 300 Mev); the meson-theoretic approach and the phenomenological approach.

Several meson theoretical potentials give a reasonable fit to the low energy data, in particular those of Levy (1952) and Gartenhaus (1955). However, these potentials fail conspicuously when an attempt is made to fit the unpolarised and polarised scattering at 100 and 150 Mev. Signell and Marshak (1957, 1958) find that this data can be fitted reasonably well by adding a phenomenological short range attractive spin-orbit potential to the Gartenhaus potential, (which already contains central and tensor components). These meson-theoretic potentials all become strongly repulsive at short distances.

Gammel and Thaler (1957) have made an extensive search for a phenomenological potential to fit the data up to 150 Mev. They looked for a potential of the Yukawa shape consisting of central, tensor and spin-orbit terms. They allowed different ranges, depths, and cut-offs in the various terms as well as in the different spin and isotopic spin states. They were able to find a set of parameters which give reasonable agreement with experiment. The central force terms considered contained an infinite repulsive core.

These alterations were made to the internucleon potential to

explain the high energy data, but meson theory indicates that one might expect the potential between two nucleons to become strongly repulsive for low energies also.

The effect of the repulsive core in the low energy region (where a potential description of nuclear forces is at least appropriate) has been studied extensively by Preston and co-workers (1955, 1956, 1957, 1958). As one of their conclusions, they find that calculations with infinite repulsive cores give results in close agreement with those for any reasonably strong repulsion. The core radius is, of course different in the two cases.

If a repulsive core potential were used in the "consistency" problem, it seems likely that the size of the three and four particle nuclei would be increased whilst the binding energy remains the same. This would help to remove the discrepancies occurring in the theoretical values for the Coulomb energy of ${}^3\text{He}$ and the maximum of the (γp) cross section for ${}^4\text{He}$.

Kikuta Morita and Yamada (1956, 1957) have already considered the effect of a hard core on the binding energies of ${}^3\text{H}$ and ${}^3\text{He}$. Using two body central forces with an infinite repulsive core, they found that the hard core interaction pushes out the wave function so that the Coulomb energy decreases to the experimental value.

In the following chapters, the effect of the hard core interaction on the α -particle wave function will be considered in an attempt to resolve some of the discrepancies discovered in the previous work.

Preliminary calculations will be performed on the two and three body nuclei.

High Energy Scattering

1.1.1. Introduction

In this chapter, a repulsive core potential will be used. The scattering properties and the low energy scattering potential will then be used in a standard variational scheme and analytical wave functions for the highest wave will give binding energies. This is the usual approach to the "resonance" problem mentioned in §6.

An arbitrary wave function ψ can be expanded in terms of energy eigenvalues

$$\psi = \sum_i a_i \psi_i \quad \text{where} \quad H \psi_i = E_i \psi_i$$

where the ψ_i form a complete orthonormal set. The energy value of H for the function ψ is then given by

$$\begin{aligned} \langle H \rangle &= \int \psi^* H \psi d\tau = \sum_i |a_i|^2 E_i \\ &= \sum_i |a_i|^2 E_i = E_0 \end{aligned}$$

CHAPTER V

Binding Energy Calculation.

1. General Formalism.

In this chapter, a repulsive core potential will be chosen to fit the deuteron properties and the two body scattering data. This potential will then be used in a standard variational calculation to find analytical wave functions for the lightest nuclei consistent with their binding energies. This is the usual approach to the "consistency" problem mentioned in 1.6.

An arbitrary wave function ψ can be expanded in terms of the energy eigenvalues

$$\psi = \sum_i a_i u_i \quad \& \quad H u_i = E_i u_i \quad \dots\dots\dots (1)$$

where the u_i form a complete orthonormal set. The expectation value of H for the function ψ is then given by

$$\begin{aligned} \langle H \rangle &= \int \psi^* H \psi \, d\tau = \sum_i E_i |a_i|^2 \\ &\geq E_0 \sum_i |a_i|^2 = E_0 \int \psi^* \psi \, d\tau \quad \dots\dots\dots (2) \end{aligned}$$

where E_0 is the lowest energy eigenvalue.

$$\text{i.e. } E_0 \leq \frac{\int \psi^* H \psi \, d\tau}{\int \psi^* \psi \, d\tau} \quad \dots\dots\dots (3)$$

The variational method consists of evaluating the integrals using a trial wave function ψ that depends on a number of parameters, and

varying these parameters until the expectation value of the energy is a minimum. The result is then an upper limit for the ground state energy of the system, which is likely to be close to the experimental value if the form of trial wave function resembles that of the eigenfunction u_0 .

The choice of the internucleon potential and of the trial nuclear wave function must now be discussed.

2. Choice of Potential and Wave Function.

To obtain a coherent view of the situation, it seems desirable to find wave functions for both ^3H and ^4He using the same approach. The method of calculation is considerably restricted by the difficulties encountered in the four-body problem.

The previous calculations on the triton by Kikuta et al. (1956) and by Feshbach and Rubinow (1955) have used a potential with an infinite repulsive core. However in the case of ^4He , this form gives rise to considerable computational difficulties. Since it has been shown by Preston (1955) that any infinite repulsive core potential is equivalent to a potential with a strongly repulsive core, it will be convenient to choose an analytical form which tends to a large finite value as the interparticle distance tends to zero. Such a potential is

$$V(r) = Ae^{-\lambda r^2} - Be^{-\mu r^2} \dots\dots\dots (4)$$

with

$$A \gg B, \quad \lambda \gg \mu$$

In the preliminary calculations discussed in this chapter only central forces will be used.

The parameters in the potential (4) will be chosen to fit the deuteron binding energy and the low energy scattering data.

Since the potential is strongly repulsive at short distances, the nuclear wave function will be considerably reduced inside the "core" but it is not necessarily zero as in the case of an infinite core.

For an infinite core, Jastrow (1955) has suggested that the binding energy of a system of particles should be calculated by the variational method using a trial wave function of the form

$$\psi = F \cdot S \quad \dots\dots\dots (5)$$

where S is the Slater determinant of plane wave functions, spin and isotopic spin functions. F is defined by

$$F = \prod_{i < j}^A f(r_{ij}) \quad \dots\dots\dots (6)$$

the product being taken over all pairs of particles. This approach has been considered by Emery (1958) and he finds that it is necessary to restrict the form of the correlation function $f(r_{ij})$. Two suitable forms which have been used are:-

$$(1) \quad \left. \begin{aligned} f(r) &= 0 & r &\leq r_c \\ f(r) &= 1 - e^{-\mu(r-r_c)} & r &\geq r_c \end{aligned} \right\} \begin{array}{l} \text{Iwamoto and Yamada (1957)} \\ \dots\dots\dots (7) \end{array}$$

$$\begin{aligned}
 (2) \quad f(r) &= 0 & r \leq r_c \\
 f(r) &= 1 - e^{-\mu(r^2 - r_c^2)} & r \geq r_c
 \end{aligned}
 \left. \vphantom{\begin{aligned} f(r) &= 0 \\ f(r) &= 1 - e^{-\mu(r^2 - r_c^2)} \end{aligned}} \right\} \text{Dabrowski (1958)}$$

..... (8)

It is convenient in the present problem to use a similar approach. The trial wave function is chosen of the form

$$\psi = F \phi \chi \quad \text{..... (9)}$$

where ϕ is one of the spatial wave functions used in the previous calculations (without the repulsive core) and χ is the spin wave function.

As neither of the above forms (7,8) of correlation function are readily dealt with, the function

$$f(r) = r^2 \quad \text{..... (10)}$$

will be used. This tends to zero as the two nucleons come close together.

The complete trial wave function will be taken of the form

$$\psi = \left\{ \sum_i a_i \phi(\mu_i) + F \sum_j b_j \phi(\mu_j) \right\} \chi = \Phi(\rho) \chi$$

..... (11)

where the a_i , b_i , μ_i are the variable parameters.

Since the potential has been chosen of the form (4) it will be consistent to choose ϕ of the Gaussian form. This will help to simplify the calculations.

$$\phi(\mu) = \exp - \mu \rho_A^2 \quad \text{..... (12)}$$

The Gaussian wave functions (12) and potential (4) were chosen primarily because of the ease in manipulation. However, in the earliest calculations on the consistency problem, it was found that a single term Gaussian wave function gave a poor result for the binding energy. It was considered that by using a sum of Gaussian terms, a reasonably accurate value for the binding energy could be obtained. This assumption has recently been verified by the work of Burke and Robertson (1957) on the low energy elastic scattering of neutrons by deuterons. These authors found that a deuteron wave function of the form

$$\psi(r) = \exp - \alpha r^2 + c \exp - \beta r^2 \dots\dots\dots (13)$$

and a Gaussian central potential, gave approximately 90% of the experimental binding energy.

3. Binding Energy Calculation. Definitions.

The calculations will be performed for the two, three and four particle nuclei.

The coordinate systems (4), (5), and (7) of Appendix A will be used.

In the three cases only symmetric spatial wave functions are considered. The corresponding spin functions are:-

$$\begin{aligned} {}^2\text{H} & \quad :- \quad \chi = \alpha(1) \alpha(2) \\ {}^3\text{H} \quad {}^3\text{He} & \quad :- \quad \chi = \frac{1}{\sqrt{2}} \alpha(3) \left\{ \alpha(1) \beta(2) - \beta(1) \alpha(2) \right\} \\ {}^4\text{He} & \quad :- \quad \chi = \frac{1}{\sqrt{2}} \left\{ \alpha(1) \beta(2) - \beta(1) \alpha(2) \right\} \\ & \quad \quad \quad \times \frac{1}{\sqrt{2}} \left\{ \alpha(3) \beta(4) - \beta(3) \alpha(4) \right\} \\ & \quad \quad \quad \dots\dots\dots (14) \end{aligned}$$

where 1, 2 denote like nucleons (as also do 3, 4)

It is assumed that the potential is of the general exchange type.

$$V(12) = V_0 (w + m M_{12} + b B_{12} + h M_{12} B_{12}) V(12) \dots\dots\dots (15)$$

with $w + m + b + h = 1$; $w + m - b - h = x$

where x is the ratio of singlet to triplet interaction between proton and neutron in an even state.

The Hamiltonian H can be written

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^A \nabla_i^2 + \sum_{\substack{i,j=1 \\ i < j}}^A V(r_{ij}) + \sum_{\substack{i,j=1 \\ i < j}} \eta \frac{e^2}{r_{ij}} \dots\dots\dots (16)$$

where $\eta = 1$ if i, j are both protons, $\eta = 0$ otherwise.

It is readily shown that, after performing the integrations over spin coordinates.

$$\int \psi^* H \psi d\tau = \int \psi^* \left\{ -\frac{\hbar^2}{m} T + N V(ij) \right\} \psi d\tau + E_{\text{coul}} \dots\dots\dots (17)$$

where T and N are given in Table XII and E_{coul} is the Coulomb energy.

In the following, the Coulomb energy will be treated as a perturbation.

TABLE XII

Number of Nucleons	2	3	4
T	∇_u^2	$\nabla_u^2 + \frac{3}{4} \nabla_r^2$	$\nabla_u^2 + \nabla_y^2 + \frac{1}{2} \nabla_R^2$

TABLE XII. (cont.)

Number of Nucleons	2	3	4
N	V_0	$\frac{3}{2} (1+x) V_0$	$3 (1+x) V_0$

Using the wave function (11) the kinetic energy term is given

by

$$\int \Phi^x T \Phi d\tau = \sum_{ij} \left\{ a_i a_j I_1 (\mu_i | \mu_j) + a_i b_j I_2 (\mu_j | \mu_i) \right. \\ \left. + a_i b_j I_3 (\mu_i | \mu_j) + b_i b_j I_4 (\mu_i | \mu_j) \right\} \\ \dots\dots\dots (18)$$

where

$$I_1 (\mu_i | \mu_j) = \int \phi^x (\mu_i) T \phi (\mu_j) d\tau$$

$$I_2 (\mu_i | \mu_j) = \int \left\{ F \phi (\mu_i) \right\}^x T \phi (\mu_j) d\tau$$

$$I_3 (\mu_i | \mu_j) = \int \phi^x (\mu_i) T \left\{ F \phi (\mu_j) \right\} d\tau$$

$$I_4 (\mu_i | \mu_j) = \int \left\{ F \phi (\mu_i) \right\}^x T \left\{ F \phi (\mu_j) \right\} d\tau$$

..... (19)

It is readily shown that

$$I_2(\mu_i | \mu_j) = I_3(\mu_j | \mu_i) \dots\dots\dots (20)$$

The potential energy term is given by two terms of the form

$$\int \Phi^x V(\lambda) \Phi \, d\tau = \sum_{ij} \left\{ a_i a_j J_1(\mu_i | \mu_j | \lambda) + 2a_i b_j J_2(\mu_i | \mu_j | \lambda) + b_i b_j J_3(\mu_i | \mu_j | \lambda) \right\} \dots\dots\dots (21)$$

where

$$J_1(\mu_i | \mu_j | \lambda) = \int \phi^x(\mu_i) V(\lambda) \phi(\mu_j) \, d\tau$$

$$J_2(\mu_i | \mu_j | \lambda) = \int \left\{ F \phi(\mu_i) \right\}^x V(\lambda) \phi(\mu_j) \, d\tau \\ = J_2(\mu_j | \mu_i | \lambda)$$

$$J_3(\mu_i | \mu_j | \lambda) = \int \left\{ F \phi(\mu_i) \right\}^x V(\lambda) \left\{ \phi(\mu_j) F \right\} \, d\tau \dots\dots\dots (22)$$

$$V(\lambda) = \exp - \lambda r^2 \dots\dots\dots (23)$$

Finally the normalisation terms are given by

$$\int \dagger^x \dagger \, d\tau = \int \Phi^x \Phi \, d\tau \dots\dots\dots (24)$$

$$= \sum_{ij} \left\{ a_i a_j J_1(\mu_i | \mu_j | 0) + 2a_i b_j J_2(\mu_i | \mu_j | 0) + b_i b_j J_3(\mu_i | \mu_j | 0) \right\}$$

since $V(0) = 1$.

The Coulomb energy term is given by

$$\begin{aligned}
 E_{\text{coul}} &= e^2 \int \phi^* \frac{1}{|\underline{u}|} \phi \, d\tau \\
 &= e^2 \sum_{ij} \left\{ a_i a_j C_1(\mu_i | \mu_j) + 2a_i b_j C_2(\mu_i | \mu_j) + b_i b_j C_3(\mu_i | \mu_j) \right\} \dots\dots\dots (25)
 \end{aligned}$$

where

$$\begin{aligned}
 C_1(\mu_i | \mu_j) &= \int \phi^*(\mu_i) \frac{1}{u} \phi(\mu_j) \, d\tau; \\
 C_3(\mu_i | \mu_j) &= \int \left\{ F\phi(\mu_i) \right\}^* \frac{1}{u} \left\{ F\phi(\mu_j) \right\} \, d\tau \\
 C_2(\mu_i | \mu_j) &= \int \left\{ F\phi(\mu_i) \right\}^* \frac{1}{u} \phi(\mu_j) \, d\tau = C_2(\mu_j | \mu_i) \\
 &\dots\dots\dots (26)
 \end{aligned}$$

The expression for the r.m.s. radius will be required later so it is given here for completeness.

$$\begin{aligned}
 R^2 &= \frac{1}{A} \int \phi^* \rho_A^2 \phi \, d\tau = \frac{1}{A} \int \phi^* \rho_A^2 \phi \, d\tau \\
 &= \frac{1}{A} \sum_{ij} \left\{ a_i a_j R_1(\mu_i | \mu_j | A) + 2a_i b_j R_2(\mu_i | \mu_j | A) + b_i b_j R_3(\mu_i | \mu_j | A) \right\} \\
 &\dots\dots\dots (27)
 \end{aligned}$$

where

$$\begin{aligned}
 R_1(\mu_i | \mu_j | A) &= \int \phi^*(\mu_i) \rho_A^2 \phi(\mu_j) \, d\tau \\
 R_3(\mu_i | \mu_j | A) &= \int \left\{ F\phi(\mu_i) \right\}^* \rho_A^2 \left\{ F\phi(\mu_j) \right\} \, d\tau
 \end{aligned}$$

$$R_2(\mu_i | \mu_j | A) = \int \left\{ \mathbb{F} \phi(\mu_i) \right\}^x \rho_A^2 \phi(\mu_j) d\tau = R_2(\mu_j | \mu_i | A) \dots\dots\dots (28)$$

When the Gaussian form

$$\phi(\mu_i) = \exp -\mu_i^2 \rho_A^2 \dots\dots\dots (12)$$

is used then

$$R_n(\mu_i | \mu_j | A) = \int \mathbb{F}^{n-1} \rho_A^2 e^{-\mu_{ij}^2 \rho_A^2} d\tau$$

where $\mu_{ij} = \mu_i + \mu_j$, $n = 1, 2, 3$ and

$$J_n(\mu_i | \mu_j | 0) = \int \mathbb{F}^{n-1} e^{-\mu_{ij}^2 \rho_A^2} d\tau$$

Therefore

$$\begin{aligned} \frac{d}{d\mu_{ij}} J_n(\mu_i | \mu_j | 0) &= - \int \mathbb{F}^{n-1} \rho_A^2 e^{-\mu_{ij}^2 \rho_A^2} d\tau \\ &= -R_n(\mu_i | \mu_j | A) \dots\dots\dots (29) \end{aligned}$$

Thus the r.m.s radius integrals may be derived directly from those for the normalisation.

4. Derivation of Potential.

In these preliminary calculations, only central forces are to be considered. The central potential is to be chosen to give the correct deuteron binding energy and to fit the two body scattering data.

The Schroedinger equation for the deuteron ground state wave function is

$$-\frac{\hbar^2}{m} \nabla^2 \psi + V(r) \psi = E \psi \quad \dots\dots\dots (30)$$

As a result of the assumption of central symmetry, the solutions may be separated according to

$$\psi_{n,l,m} = f_{nl}(r) Y_l^m(\theta, \phi) \quad \dots\dots\dots (31)$$

The solution of interest corresponds to the lowest energy eigenvalue i.e. to the s-state for a central potential $l=0$.

Writing

$$f_{10}(r) = \frac{u(r)}{r} \quad \dots\dots\dots (32)$$

the Schroedinger equation can be reduced to

$$\frac{d^2 u}{dr^2} - k^2 u = U(r) u(r) \quad \dots\dots\dots (33)$$

where $k^2 = \frac{m}{\hbar^2} E_d$, $U(r) = \frac{m}{\hbar^2} V(r)$ and E_d is the binding energy of the deuteron.

The form of the potential chosen is

$$V(r) = A e^{-\lambda r^2} - B e^{-\mu r^2} \quad \dots\dots\dots (4)$$

$$= B e^{-\mu r^2} \left\{ X e^{-\eta r^2} - 1 \right\} \quad \dots\dots\dots (34)$$

where B is positive. The value of X has been chosen arbitrarily to be 25. This ensures that the core is strongly repulsive.

The core radius r_c is defined by

$$V(r_c) = 0 \quad \dots\dots\dots (35)$$

Several values of the core radius have been chosen. The value of η is defined by the choice of X and of r_c . The pairs of values used are shown in Table XIII.

TABLE XIII.

$r_c \cdot 10^{-13} \text{ cm}$	0	0.2	0.4	0.6	0.8
η	∞	80	20	9	5

It is now necessary to solve the equation (33) for a potential with a given core radius r_c and a given range μ ; so as to find the eigenvalue for the depth B consistent with the deuteron binding energy.

The equation (33) is subject to the boundary conditions

$$u(0) = 0 \quad , \quad u(r) \sim e^{-kr} \quad \dots\dots\dots (36)$$

Two methods have been used to solve this eigenvalue problem.

The first method involves a step-by-step integration of the equation from $r = \infty$ to $r = 0$. The parameter B is chosen arbitrarily and starting from the asymptotic form at some finite but very large distance, the numerical integration can be carried in to the origin.

The method VII of the paper by Fox and Goodwin (1949) was used. Since the parameter B has been chosen arbitrarily, the value of $u(0)$ will not satisfy the boundary condition (36). After intelligent observation of the calculated value of $u(0)$, a new choice of B is made (for the same μ) and the procedure is repeated. A number of repetitions determines the value of B, for a given μ , which satisfies the boundary condition $u(0) = 0$ to the desired approximation.

This method is rather tedious and it is useful to consider a second method. The equation (33) may be transformed to an integral equation incorporating the boundary conditions (36).

$$\text{Writing } U(r) = B^1 W(r) \quad , \quad B^1 = \frac{m}{h^2} B,$$

$$W(r) = e^{-\mu r^2} (X e^{-\eta r^2} - 1)$$

then

$$u(r) = \frac{B^1}{k} \left\{ \sinh kr \int_{\infty}^r \exp(-k\zeta) W(\zeta) u(\zeta) d\zeta \right. \\ \left. - \exp(-kr) \int_0^r \sinh k\zeta W(\zeta) u(\zeta) d\zeta \right\} \\ \dots\dots\dots (37)$$

This integral equation is then most readily solved by an iteration process.

A series of iterated functions is constructed from a properly chosen trial wave function $u_0(r)$ using

$$u_{n+1}(r) = \frac{B^1 n}{k} \left\{ \sinh kr \int_{\infty}^r \exp(-k\zeta) W(\zeta) u_n(\zeta) d\zeta - \right.$$

$$- \exp(-kr) \int_0^r \sinh k\zeta W(\zeta) u_n(\zeta) d\zeta \} \dots\dots\dots (38)$$

Then since $u(r) \sim \exp -kr$, it is easily shown that the n^{th} approximation to the eigenvalue is

$$B_n^{-1} = \frac{-k}{\int_0^\infty \sinh kr W(r) u_n(r) dr} \dots\dots\dots (39)$$

It was found that the best procedure in the present problem, was to use a combination of the above two methods. Using a suitable trial solution, the iteration method is applied to find an approximate value of the eigenvalue B . This value of B is then used as a first approximation in the step-by-step integration method

The eigenvalue problem has been solved for a range of values of μ for each core radius. The results are shown in Figure V.

For each set of values (B, μ, r_c) , the triplet scattering length a_t has been calculated by integrating the zero energy equation

$$\frac{d^2 u}{dr^2} = B W(r) u(r) \dots\dots\dots (40)$$

outwards from the origin. The intercept on the r -axis gives the value of a_t . The values of a_t accurate to approximately 2% are given in Figure VI.

As the binding energy calculations depend on the exchange nature of the force only through the parameter x , it will be sufficient at this stage simply to calculate this parameter, the ratio of the

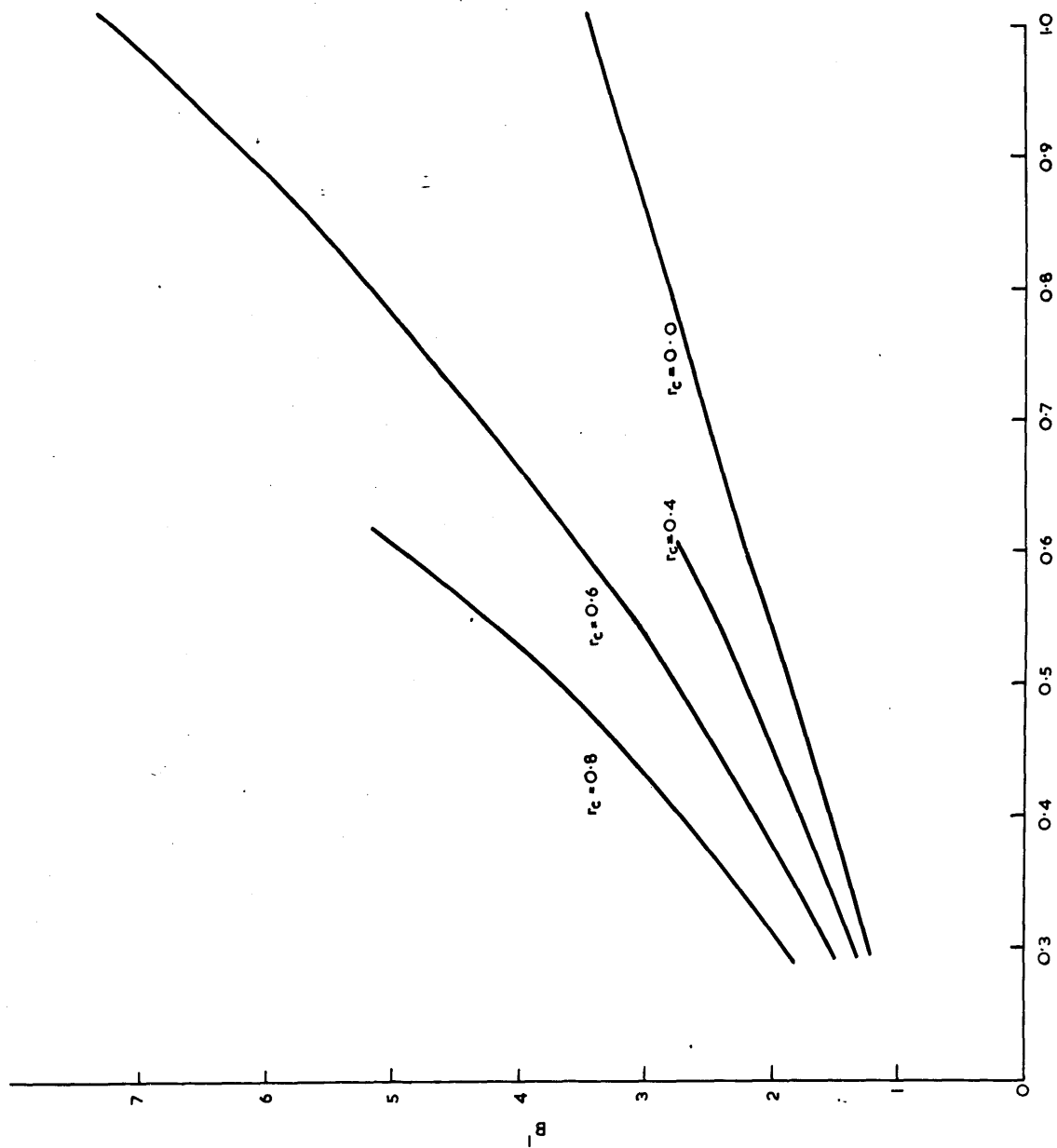


FIGURE 5. WELL DEPTH B^1 $\frac{m}{\hbar^2} V(r) = B^1 e^{-\mu r^2} (x e^{-\gamma r^2} - 1)$

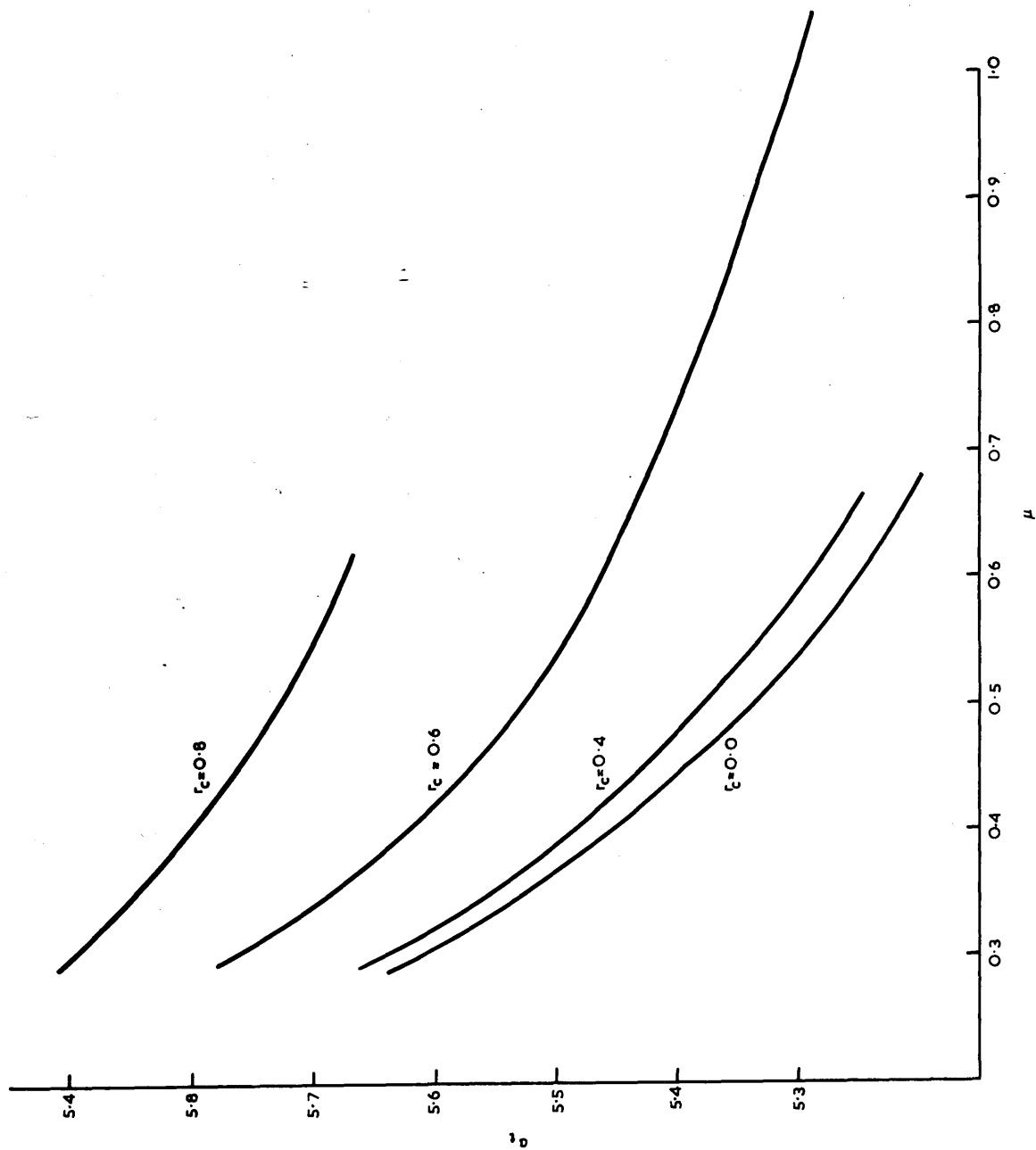


FIGURE 6. TRIPLET SCATTERING LENGTH α_t

singlet and triplet interactions between proton and neutron in an even state. x is then chosen as the parameter required in the zero energy equation for the singlet state

$$\frac{d^2 u}{dr^2} = B x W(r) u(r) \quad \dots\dots\dots (41)$$

to give the experimental value for the singlet scattering length a_s . However it is found that x is not very sensitive to the potential shape, and its value has been chosen as 0.6 in all the present calculations.

It should be noted here that the experimental values for the triplet and singlet scattering lengths are

$$a_t = 5.38 \times 10^{-13} \text{ cm} \quad a_s = -23.7 \times 10^{-13} \text{ cm} \quad \dots\dots\dots (42)$$

In future all lengths will be given in units of 10^{-13} cms.

5. Evaluation of the Integrals.

The various integrals defined in §3, have been evaluated for the two, three and four particle nuclei using the Gaussian wave functions and the Gaussian type potential introduced in the previous section i.e. the integrals have been evaluated with

$$\phi(\mu) = \exp - \mu \rho_A^2 \quad ; \quad V(\lambda) = \exp - \lambda u^2$$

The results for the two and three particle nuclei are tabulated below. Those for the four body case are given in Appendix D.

Kinetic Energy Integrals.

A	$I_1 (\mu_i \mu_j)$	$I_2 (\mu_i \mu_j)$	$I_4 (\mu_i \mu_j)$
2	$\frac{-6\pi^{3/2} \mu_i \mu_j}{\mu_{ij}^{5/2}}$	$\frac{3\pi^{3/2} \mu_j (2\mu_j - 3\mu_i)}{\mu_{ij}^{7/2}}$	$\frac{3\pi^{3/2} \{6\mu_{ij}^2 - 35\mu_i \mu_j\}}{2\mu_{ij}^{9/2}}$
3	$\frac{-6\pi^3 \mu_i \mu_j}{\sqrt{3} \mu_{ij}^4}$	$\frac{2\sqrt{3} \pi^3 \mu_j (\mu_j - \mu_i)}{3\sqrt{3} \mu_{ij}^7}$	$\frac{35\pi^3 \{11\mu_{ij}^2 - 57\mu_i \mu_j\}}{9\sqrt{3} \mu_{ij}^{10}}$

..... (43)

where $\mu_{ij} = \mu_i + \mu_j$.

Potential Energy Integrals.

A	$J_1 (\mu_i \mu_j \lambda)$	$J_2 (\mu_i \mu_j \lambda)$	$J_3 (\mu_i \mu_j \lambda)$
2	$\frac{\pi^{3/2}}{\eta_{ij}^{3/2}}$	$\frac{3\pi^{3/2}}{2\eta_{ij}^{5/2}}$	$\frac{15\pi^{3/2}}{4\eta_{ij}^{7/2}}$
3	$\frac{\pi^3}{(2\mu_{ij} \eta_{ij})^{3/2}}$	$\frac{15\pi^3}{2^5 (2\mu_{ij})^{3/2} \eta_{ij}^{5/2}}$	$\frac{105\pi^3}{2^{10} (2\mu_{ij})^{3/2} \eta_{ij}^{7/2}}$

..... (44)

where $\eta_{ij} = \mu_i + \mu_j + \lambda$ for $A = 2$
 $\dots\dots\dots (45)$
 $= \frac{3}{2} (\mu_i + \mu_j) + \lambda$ for $A = 3$

and $X = \left\{ \frac{3}{\mu_{ij}^2} + \frac{1}{\mu_{ij}} \eta_{ij} + \frac{7}{4} \eta_{ij}^2 \right\}$

$Y = \left\{ \frac{9.5.3}{\mu_{ij}^4} + \frac{7.5.2}{\mu_{ij}^3} \eta_{ij} + \frac{29.9}{\mu_{ij}^2} \eta_{ij}^2 + \frac{11.9}{2 \mu_{ij}} \eta_{ij}^3 \right.$
 $\left. + \frac{13.11.9}{16 \eta_{ij}^4} \right\}$

Normalisation Integrals.

A	$J_1 (\mu_i \mu_j 0)$	$J_2 (\mu_i \mu_j 0)$	$J_3 (\mu_i \mu_j 0)$
2	$\frac{\pi^{3/2}}{\mu_{ij}^{3/2}}$	$\frac{3 \pi^{3/2}}{2 \mu_{ij}^{5/2}}$	$\frac{15 \pi^{3/2}}{4 \mu_{ij}^{7/2}}$
3	$\frac{\pi^3}{3^{3/2} \mu_{ij}^3}$	$\frac{25 \pi^3}{3^{7/2} 2 \mu_{ij}^6}$	$\frac{19.7.5 \pi^3}{2.3^9 2 \mu_{ij}^9}$

$\dots\dots\dots (46)$

Mean Square Radius Integrals.

A	$\frac{1}{A^2} R_1 (\mu_i \mu_j)$	$\frac{1}{A^2} R_2 (\mu_i \mu_j)$	$\frac{1}{A^2} R_3 (\mu_i \mu_j)$
2	$\frac{3 \pi^{3/2}}{8 \mu_{ij}^{5/2}}$	$\frac{15 \pi^{3/2}}{16 \mu_{ij}^{7/2}}$	$\frac{105 \pi^{3/2}}{32 \mu_{ij}^{9/2}}$
3	$\frac{\pi^3}{3^{5/2} \mu_{ij}^4}$	$\frac{25 \pi^3}{3^7 \mu_{ij}^7}$	$\frac{19 \cdot 7 \cdot 5 \pi^3}{2 \cdot 3^9 \mu_{ij}^{10}}$

..... (47)

Coulomb Energy Integrals.

A	$(\mu_i \mu_j)$	$(\mu_i \mu_j)$	$(\mu_i \mu_j)$
3	$\frac{\sqrt{2} \pi^{5/2}}{3 \mu_{ij}^{5/2}}$	$\frac{61 \pi^{5/2}}{3^3 \cdot 2^{5/2} \mu_{ij}^{11/2}}$	$\frac{15089 \pi^{5/2}}{3^5 \cdot 2^{9/2} \mu_{ij}^{17/2}}$

..... (48)

6. Variational Calculation.

Using the integrals given in the previous section the variational calculations for the binding energy of the lightest nuclei were performed. To test the method the case of the deuteron was considered first.

(a) Deuteron Calculation.

Whilst deriving the potential for the two body problem(44), the deuteron wave function was found corresponding to each potential considered. This wave function could be fitted approximately by a two term function of the form (11). This analytical wave function should then be used as a trial wave function in the variational calculation.

So that the effect of introducing the repulsive core may be fully appreciated, a large core radius $r_c = 0.6$ was chosen. The deuteron wave functions in this case fall almost to zero at the origin and hence the modified Gaussian wave function may be tested with this choice of core radius.

The first calculations were performed using a very deep well which corresponds to a triplet scattering length $a_t = 5.3$, slightly less than the experimental value. The parameters used are given by

$$\frac{m}{\hbar^2} V(r) = B^1 e^{-\mu r^2} (X e^{-\eta r^2} - 1) \dots\dots\dots (49)$$

$$B^1 = 7.275 \quad \mu = 1.0 \quad \eta = 9.0 \quad X = 25 \quad a_t = 5.3$$

It was found that the function

$$\phi_2^D(r) = r^2 \left[e^{-\alpha r^2} + c e^{-\beta r^2} \right] \dots\dots\dots (50)$$

with

$$c = 0.1 \quad \alpha = 1.5 \quad \beta = 0.25$$

gave an approximate fit to the exact numerical wave function. Using this function as a first trial function, the variational calculation was performed minimising with respect to c . It was found that with the parameters $(\alpha\beta c) = (1.5, 0.25, 0.1)$, the energy was a minimum, being + 3.10 MeV. Thus with this wave function, the deuteron is unbound.

The values of (α, β) were then altered and the calculation repeated. This was carried out several times until the energy was minimised with respect to the variation of all three parameters (α, β, c) . The final minimum was found with the set of parameters $(\alpha, \beta, c) = (1.1, 0.22, 0.1)$. This corresponds to an energy of + 1.17 MeV. Hence with a trial wave function of the form (50) it is impossible to find a set of parameters which gives a bound deuteron for the chosen potential.

It should be noted here that the variation in the energy is relatively much less sensitive to the variation of c than to the variation of α and β .

The wave function (50) giving the minimum energy is compared in Figure VII with the exact numerical wave function found whilst deriving the potential. It is seen that the exact wave function rises more rapidly to the maximum but then falls

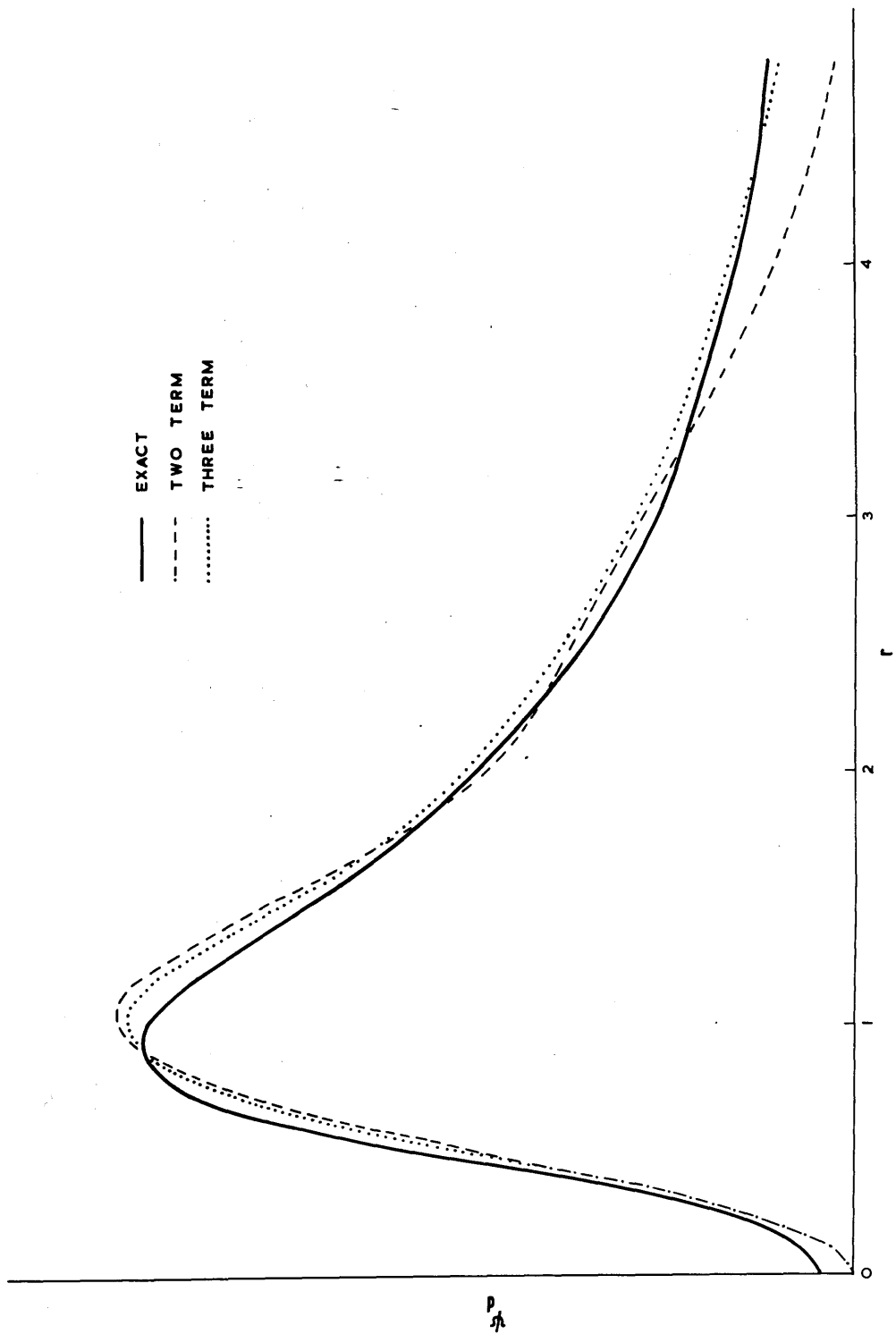


FIGURE 7. DEUTERON WAVE FUNCTIONS $\psi_d(r)$

off much more slowly at large values of r . The trial wave function was fitted approximately for the smaller values of r ($r < 3$) and the resultant variational wave function is a compromise between the correct form at small r (effect of the core) and the correct asymptotic form. This deficiency may be remedied, at least in part, by adding a third term to the wave function (50) i.e.

$$\phi_3^D(r) = r^2 \left[e^{-\alpha r^2} + c e^{-\beta r^2} + d e^{-\gamma r^2} \right]$$

..... (51)

As a first trial, the function ϕ_3 was used with (α, β, γ) chosen as those values which gave the minimum energy for ϕ_2 . A value for γ was then chosen and the variational calculation was carried out minimising with respect to d . The minimum energy was found for $(\gamma d) = (0.04, 0.0045)$ being -1.01 MeV. Thus the three term wave function (51) gives a bound deuteron a considerable improvement on (50).

The effect of varying c and d simultaneously was investigated keeping (α, β, γ) constant, and a minimum was found for $(c, d) = (0.091, 0.0044)$ giving an energy of -1.02 MeV. As this is a negligible improvement on the previous value, it will be sufficient in future to fix c as the value which minimises the energy given by ϕ_2 , and then to minimise with respect to d .

To find the minimum value of the energy corresponding to the wave function ϕ_3 , it is necessary to vary all five parameters ($\alpha, \beta, \gamma, c, d$) simultaneously. This entails a very considerable amount of computation and has only been done approximately using a desk machine. To do the minimisation completely, the calculation should be programmed for an electronic computer. The approximate minimum was found with the set of parameters ($\alpha\beta\gamma cd$) = (1.2, 0.25, 0.05, 0.108, 0.006). The minimum energy was - 1.21 MeV.

This deuteron wave function giving the best binding energy is shown in Figure VII where it is compared with the best ϕ_2 and the exact numerical wave function. This wave function gives a better fit than ϕ_2 for large r but makes little change near the origin.

It should be noted that the addition of a third term to the wave function causes the values of both α and β to increase slightly, shifting the maximum of the wave function to smaller r as is required to fit the numerical wave function.

Burke and Robertson (1957) in deriving deuteron wave functions for a calculation on the low energy elastic scattering of neutrons by deuterons, used a Gaussian potential

$$V(r) = -B e^{-\mu r^2} \dots\dots\dots (52)$$

and a wave function of the form

$$\phi_D(r) = e^{-\alpha r^2} + c e^{-\beta r^2} \dots\dots\dots (53)$$

They find that the minimum energy obtained in their variational calculation increases as the well depth increases. The maximum well depth of the potential used in the present calculations is 150 MeV i.e. a very deep well, and it is therefore in agreement with the work of Burke and Robertson, that the two term wave function ϕ_2 gives a poor value of the binding energy.

It would be desirable for completeness to perform the variational calculation for the deuteron using the trial wave function ϕ_3 for a set of different potentials with various ranges and core radii. However as was seen above, this entails an almost insuperable amount of calculating for desk computation. In all probability a fourth term will be needed in the trial wave function. Since the main object of this investigation is to find wave functions for the triton and for the alpha-particle, the deuteron problem will be left at this stage and the triton problem considered.

(b) Triton Calculation.

In all earlier calculations on the triton using central forces, it has been found that, if the potential is chosen to fit the deuteron binding energy and the two body scattering data, then the binding energy of the triton is in excess of the experimental value. In the present calculation, the triton binding energy is found first for a series of Gaussian potentials (52) with no core.

A trial wave function of the form

$$\phi_2^T(\rho) = e^{-\alpha\rho^2} + c e^{-\beta\rho^2} \dots\dots\dots (54)$$

was chosen and the variational calculation was performed for the triton assuming the ratio of singlet to triplet interaction to be $x = 0.6$.

As a first choice c was chosen to be zero. A minimum energy of -6.79 MeV was found for $\mu = 0.3$ $B = 51.54$ MeV and $\alpha = 0.085$. This energy is decreased to -7.97 MeV by the addition of a second term in the wave function. The corresponding set of parameters is $(\alpha\beta c) = (0.09, 0.03, 0.09)$. By the addition of further terms in the wave function it will be possible to increase the binding energy to the experimental value of 8.3 MeV. The potential corresponds to a triplet scattering length $a_t = 5.6$. In the calculations using repulsive core potentials, the interaction will be chosen to fit the deuteron binding energy and also this triplet scattering length (rather than the experimental one). This gives a suitable equivalent central potential for the study of the triton.

Preliminary calculations were carried out with the repulsive core potentials with $r_c = 0.2$ and $r_c = 0.4$ corresponding to $a_t = 5.6$. It was found that these potentials change the binding energy insignificantly as compared with the zero core case, and leave the wave function almost unaltered. Attention will therefore be concentrated on the core radius $r_c = 0.6$. The required

potential is given by (49) with the parameters

$$B^1 = 2.46 \quad \mu = 0.45 \quad \eta = 9.0 \quad X = 25 \quad a_t = 5.6$$

..... (55)

Using a trial wavefunction of the form

$$\phi T(\rho) = e^{-\alpha \rho^2} + c F e^{-\beta \rho^2} \quad \text{..... (56)}$$

it was found that the triton was just bound with an energy of -0.13 MeV for the Gaussian wave function $(\alpha c) = (0.04, 0)$. With the two term wave function (56) the minimum energy was found by a similar trial-and-error procedure to that used in the two-body case. The minimum energy was found to be -3.0 MeV with the set of parameters $(\alpha \beta c) = (0.05, 0.5, 2.11)$. To improve on this energy, a three term wave function is required.

The calculations were repeated for a second potential, that consistent with the experimental scattering length $a_t = 5.4$. The corresponding parameters are

$$B^1 = 4.75 \quad \mu = 0.75 \quad \eta = 9.0 \quad X = 25 \quad a_t = 5.4$$

..... (57)

With a two term trial wave function of the form (56) it was found impossible to obtain a bound triton. This may be explained as follows.

The decrease in the scattering length from 5.6 to 5.4 entails

a considerable increase in the depth of the potential and also doubles the height of the core. This implies that the wave function is greatly reduced inside the core. A more satisfactory trial wave function for this potential (57) would then be of the form

$$\phi^T(\rho) = F \left\{ e^{-\alpha\rho^2} + c e^{-\beta\rho^2} \right\} \dots\dots\dots (58)$$

(c) Discussion.

In the deuteron and triton calculations it has been seen that to obtain approximately the correct binding energy, it will be necessary to use a minimum of three terms in the trial wave function. For the potential considered in the deuteron problem, a three term wave function gave only 53% of the binding energy.

To perform the variational calculation, the minimisation has to be performed with respect to at least five parameters. The energy is fairly sensitive to the parameters occurring in the exponential but is relatively insensitive to the linear parameters. In addition, the combination of terms in the trial wave function depends on the magnitude of the core. As the integrals involved are complicated, the minimisation cannot be done by an analytical method but can only be carried through by a trial-and-error search. This search can only be made satisfactorily using an electronic computer since it is desirable to repeat the variational calculation several times using a series

of potentials with various ranges and core radii, for each of the three nuclei considered.

Because of these complexities, no calculations have yet been performed on the alpha-particle.

Before discussing the relative advantages and disadvantages of this approach to the problem of finding suitable analytic wave functions for the lightest nuclei, the application of the repulsive core wave functions (11) to the study of the photodisintegration cross section will be considered.

7. Photodisintegration of ${}^3\text{H}$ and ${}^3\text{He}$.

The photodisintegration cross section was derived in (III (6)) as

$$\sigma_{\text{dis}} = \frac{2}{3} \left(\frac{M_e^1}{\hbar^2} \right) kK |M_{ab}|^2 \dots\dots\dots (59)$$

where

$$M_{ab} = \sum_{i=1}^Z \int \phi_b^* \mathcal{E}_i \phi_a d\tau \dots\dots\dots (60)$$

For the reaction ${}^3\text{H}({}^3\text{H}(\gamma n){}^2\text{H})$ the electric dipole cross section is then given by (59), (60) with $Z = 2$ and the reduced mass $M^1 = \frac{2}{3} m$ where m is the nucleon mass.

The coordinate system (5) of Appendix A will be used throughout

$$\underline{u} = \underline{r}_2 - \underline{r}_1 \qquad \underline{r} = \underline{r}_3 - \frac{1}{2} (\underline{r}_1 + \underline{r}_2) \dots\dots\dots (61)$$

where 1 denotes the proton, 2, 3 the neutrons. Then

$$M_{10} = - \int \phi_b^x \left(\frac{1}{2} u_z + \frac{1}{2} r_z \right) \phi_a \, d\tau \quad \dots\dots\dots (62)$$

The initial triton wave function ϕ_a may be written

$$\phi_a = \psi_T = \phi_T(1\overline{23}) \, 6_T(1\overline{23}) \quad \dots\dots\dots (63)$$

If the wave function is chosen to be spatially symmetric in all three nucleons, the spin wave function must be antisymmetric in the neutrons 2,3 and hence

$$6_T(1\overline{23}) = \frac{1}{\sqrt{2}} \alpha(1) \left\{ \alpha(2) \beta(3) - \beta(2) \alpha(3) \right\} \quad \dots\dots\dots (64)$$

The final state, deuteron plus neutron, may be represented by a resonating group wave function

$$\phi_b = \frac{1}{\sqrt{2}} (1-P_{23}) \, \phi_D(12) \, F(3) \, 6_F(123) \quad \dots\dots\dots (65)$$

where

$$6_F(123) = \frac{1}{\sqrt{6}} \left\{ \alpha(3) [\alpha(1) \beta(2) + \beta(1) \alpha(2)] - 2 \beta(3) \alpha(1) \alpha(2) \right\} \quad \dots\dots\dots (66)$$

ϕ_T and ϕ_D are the symmetric spatial wave functions for the triton and deuteron respectively.

Introducing the wave functions ϕ_a and ϕ_b into M_{ab} (60) gives after summing over the spins

$$M_{ab} = - \sqrt{\frac{3}{2}} \int \phi_T(123) \, \phi_D(12) \, F(3) \, \left(\frac{1}{2} u_z + \frac{1}{2} r_z \right) \, du \, dr \quad \dots\dots\dots (67)$$

In the Born approximation, the outgoing neutron will be represented by a plane wave,

$$F(\underline{z}) = e^{i \underline{k} \cdot \underline{r}} \dots\dots\dots (68)$$

For the reaction ${}^3\text{He}(\gamma p){}^2\text{H}$ the matrix element is

$$M_{ab} = \int \phi_b^* (\underline{z}_2 + \underline{z}_3) \phi_a \, d\tau \dots\dots\dots (69)$$

where 2,3 denote the protons and 1 the neutron. The electric dipole operator is then

$$\underline{z}_2 + \underline{z}_3 = \frac{1}{3}\underline{r}_Z + \frac{1}{2}\underline{u}_Z = -\underline{z}_1 \dots\dots\dots (70)$$

Hence the matrix element for this reaction is identical with that for the ${}^3\text{H}(\gamma n){}^2\text{H}$ reaction when the Coulomb repulsion is ignored.

The wave functions discussed in the previous few sections will be used to evaluate the matrix element M_{ab} in (67).

The deuteron wave function used is

$$\Phi_D(\underline{u}) = \sum_i a_i \phi^D(\underline{\mu}_i) + F^D \sum_j b_j \phi^D(\underline{\mu}_j) \dots\dots\dots (71)$$

whilst the triton wave function is

$$\Phi_T(\rho) = \sum_i c_i \phi^T(\lambda_i) + F^T \sum_j d_j \phi^T(\lambda_j) \dots\dots\dots (72)$$

where

$$F^D = u^2 \quad F^T = \frac{1}{16} \left\{ u^6 + 3u^4 r^2 + 16u^2 r^4 - 16u^2 (\underline{u} \cdot \underline{r})^2 \right\}$$

$$\phi^D(\mu) = \exp -\mu u^2 \quad \phi^T(\lambda) = \exp -\lambda \rho^2$$

..... (73)

and

$$\rho^2 = 2r^2 + \frac{3}{2} u^2$$

The matrix element consists of a sum of terms of the form

$$\begin{aligned} M_1(\mu|\lambda) &= \int e^{-\mu u^2} e^{-\lambda \rho^2} r_{\underline{z}} e^{i \underline{k} \cdot \underline{r}} d\underline{u} d\underline{r} \\ M_2(\mu|\lambda) &= \int F^D e^{-\mu u^2} e^{-\lambda \rho^2} r_{\underline{z}} e^{i \underline{k} \cdot \underline{r}} d\underline{u} d\underline{r} \\ M_3(\mu|\lambda) &= \int F^T e^{-\mu u^2} e^{-\lambda \rho^2} r_{\underline{z}} e^{i \underline{k} \cdot \underline{r}} d\underline{u} d\underline{r} \\ M_4(\mu|\lambda) &= \int F^D F^T e^{-\mu u^2} e^{-\lambda \rho^2} r_{\underline{z}} e^{i \underline{k} \cdot \underline{r}} d\underline{u} d\underline{r} \\ &\quad \dots\dots\dots (74) \end{aligned}$$

with the corresponding integrals $N_n(\mu|\lambda)$ with $r_{\underline{z}}$ replaced by $u_{\underline{z}}$.

It is easily shown that

$$N_n(\mu|\lambda) \equiv 0 \quad n = 1, 2, 3, 4. \quad \dots\dots\dots (75)$$

It is convenient to introduce the integral

$$J(l m n; \alpha \beta) = \int e^{i \underline{k} \cdot \underline{r}} u^{2l} r^{2m} (\underline{u} \cdot \underline{r})^{2n} e^{-\alpha u^2} e^{-\beta r^2} r_{\underline{z}} d\underline{u} d\underline{r}$$

..... (76)

This integral is required for $n = 0, 1$.

$$J(1\ m\ 0; \alpha\beta) = \pi^{5/2} k \frac{\Gamma\left(\frac{2l+3}{2}\right)}{\alpha^{(2l+3)/2}} (-1)^m \frac{d^m}{d\beta^m} f(\beta)$$

$$\text{where } f(\beta) = \frac{1}{\beta^{5/2}} \exp - k^2/4 \beta \dots\dots\dots (77)$$

and

$$J(1, m, 1; \alpha\beta) = \frac{1}{3} J(1+1, m+1, 0; \alpha\beta)$$

Using this integral, the values of $M_n(\mu|\lambda)$ can be readily obtained.

In the previous section, a variational calculation was described, which was used to find analytical wave functions for the deuteron and the triton. Once these wave functions have been obtained, it is a very straight-forward calculation to evaluate the matrix element M_{ab} using the integrals $M_n(\mu|\lambda)$ and $J(1\ m\ n; \alpha\beta)$ and hence to evaluate the photodisintegration cross section.

No numerical results have yet been obtained since it was found impossible by desk calculation to find a deuteron and a triton wave function giving the correct binding energies. Until these wave functions have been calculated by a computer, it would serve no worthwhile purpose to calculate the cross section.

This consideration of the photodisintegration cross section can be extended to the case of the alpha particle, although the calculation increases greatly in complexity.

CHAPTER VI.

Final Conclusions.

At this point, it is advisable to summarise the results of Chapter V. It has been found that the amount of computation involved in the present approach is enough to warrant machine calculation. Since this approach was intended primarily as a preliminary calculation, it is necessary to re-assess the position before embarking on a large scale machine calculation.

The main object of this investigation was to find a wave function for the alpha-particle which could be used to calculate the photo-disintegration cross section. This requirement has considerably restricted the choice of potential and wave function employed. Moreover, even with the simple Gaussian and modified Gaussian forms used, the amount of calculation required in the study of the binding energy and of the photodisintegration is already very considerable. The algebra involved in the derivation of the (γp) cross section for the alpha particle will be extremely long and tedious.

The Gaussian wave functions used, have a poor asymptotic form. This has been shown clearly in the binding energy calculations on the deuteron. There it is seen that the variational calculation has led to a wave function which makes a compromise between two effects, one at small r due to the core, the other adjusting the wave function to

the proper asymptotic form at large r . It is therefore desirable to choose a wave function with the correct asymptotic form, for then all improvements in the form of the wave function will be at small r in the region of the repulsive core. This requirement would be met in the case of the deuteron if the Exponential form wave function, was chosen. This form would also be an improvement in the case of the three and four body nuclei as it has already been shown that the Exponential wave function has a reasonable asymptotic form for these nuclei.

If this modification was made to the calculation, it is probable that the number of terms required in the wave function to give the experimental binding energy would be reduced. However, the algebra occurring in the binding energy calculation would become considerably more complex, necessitating the numerical evaluation of the integrals in the variational calculation.

The choice of potential was also restricted because it was desired that the alpha particle be studied. If the calculations had been restricted to the three body nuclei, it would have been possible to evaluate the integrals using a potential with an infinite repulsive core. Extensive sets of parameters for such potentials are available. The most important of these are the set of parameters given by Gammel and Thaler (1957) which fit the two body data up to 310 MeV, and the sets given by Biedenharn, Kalos and Blatt (1958) which fit the low energy two body data. These potentials include both central and tensor

components and in the case of the Gammel-Thaler potential a spin-orbit term as well.

In the present calculations, the effect of the tensor force has been introduced only by using an equivalent central potential. This is a rather unsatisfactory approach which is only reasonable if the calculations can be done for a purely central force with a great reduction in the computation, enabling the physical picture to be seen more clearly. In the case of internucleon potentials with no repulsive core, it is known that the addition of a tensor component alters the binding energies of the triton and the alpha particle obtained in variational calculations quite appreciably. Since an electronic computer is required to perform the central force calculations for both the deuteron and the triton, it seems more logical to find ways of modifying the approach so that the tensor force can be introduced from the beginning. If this could be done, then a large scale numerical calculation would be very worthwhile.

Since this work was completed, Blatt and Derrick (1958) have published a report on some calculations on repulsive core forces in the triton which cast doubt on the validity of the equivalent central potential.

Writing the wave equation as

$$(T + \lambda V) \phi = E \phi$$

they derived a variational expression which gives an upper bound for

the lowest eigenvalue λ . Two sets of potentials were considered.

(a) The set of potentials given by Biedenharn, Kalos and Blatt.

These potentials show an infinite repulsive core and both central and tensor components. Only the central part is used in the calculations.

(b) The set of equivalent central potentials of Kikuta et al which fit the deuteron binding energy and the two body scattering data but do not, of course, give the deuteron quadrupole moment.

The minimum value of λ is found for both sets of potentials with various core radii. If λ is less than unity, the corresponding potential gives too much binding. For the set (a) it is found that λ is always greater than unity, the remainder of the binding being contributed by the tensor component. Also the force strength λ decreases with increasing core radius r_0 for a given triplet central well depth. This means that for the actual potential, the binding energy increases with increasing core radius r_0 . The opposite result is found for the set (b). This time λ increases with increasing core radius. Also for small r_0 , λ is less than unity indicating that the potential is unacceptable. This is the well-known result that a central force of zero core radius which binds the deuteron properly, gives too much binding for the triton. Because of this contradiction between the results given by the sets (a), (b), the concept of an equivalent central potential must be viewed with suspicion.

This investigation of the wave functions of the lightest nuclei has shown that a wave function which gives a good fit to the binding energy using the variational approach, need not necessarily be a good approximation to the wave function. It was not until the size of the alpha particle was determined by the electron scattering experiments at Stanford, that it was realised that the variationally derived wave functions gave much too concentrated an alpha particle. This fact had lain latent for some time, in the fact that the Coulomb energy of ${}^3\text{He}$ as found in the variational calculations, was considerably larger than the experimental value. Thus, it is seen that the binding energy is not a good criterion in itself for choosing a wave function, but this must be coupled with some other property such as the root mean square radius or the Coulomb energy. The insufficiency of the binding energy as a criterion is partly due to the fact that the binding energy is the difference between two large energies (~ 50 MeV), the kinetic and potential energies.

Since many of the difficulties encountered in the present calculations are due to necessity of choosing simple analytical forms in any calculations on the alpha particle, it is logical to restrict future calculations to the case of the three body nuclei, when a better choice of potential and wave function may be made.

However, there is a considerable lack of experimental data and additional experiments would be of great assistance in guiding the

theoretical approach. The experiments of Cranberg (1958) on the photodisintegration of ${}^3\text{He}$ should be extended and improved, particularly with regards distinguishing the two and three body break-up processes. This could probably be done best by obtaining the cross section for the ${}^3\text{He}(\gamma p){}^2\text{H}$ reaction by studying the inverse reaction ${}^2\text{H}(p\gamma){}^3\text{He}$. The cross sections could then be compared by the use of the detailed balance theorem. This is only possible because there are no excited states in ${}^3\text{He}$. A knowledge of the r.m.s radius is also desirable. This can be obtained either directly by electron scattering or indirectly from the bremsstrahlung-weighted cross section.

The ideal calculation to perform to clear up the discrepancy between the size of the lightest nuclei as obtained experimentally and theoretically, is a variational calculation on the binding energy of ${}^3\text{H}$ and ${}^3\text{He}$ using a fully realistic potential with central and tensor components. Its aim should be to fit both the binding energy of ${}^3\text{H}$ and the Coulomb energy of ${}^3\text{He}$. Since the calculations would be restricted to the three body nuclei, it would be possible to use infinite repulsive cores in the potential and also to ensure that the wave function has a reasonable asymptotic form.

This type of calculation has been performed by Kikuta et al using central forces. As has already been mentioned, it is desirable to extend this to include the effect of the tensor force. The potentials of Gammel and Thaler and of Biedenharn et al could be used.

The triton wave function will be most readily dealt with if written in the form

$$\psi_T = \psi(R) \prod_{\substack{ij=1 \\ i < j}}^3 f(r_{ij}) \left\{ |S\rangle + \lambda |p\rangle + \mu |d\rangle + \dots \right\} \quad (1)$$

where

$$R = r_{12} + r_{23} + r_{31}$$

The p and d state are introduced in the Schwinger-Gerjuoy operator formalism as was done by Hu and Hsu (1951). The correlation function can then be suitably chosen. $\psi(R)$ can then be found either by using the equivalent two body method (cf. Feshbach and Rubinow 1955) or else by the variational method as used by Kikuta et al. These last authors used a trial wave function of the form (1) but with the S state only

$$\begin{aligned} \psi &= \prod_{\substack{ij=1 \\ i < j}}^3 \left\{ e^{-\mu(r_{ij} - r_0)} - e^{-\nu(r_{ij} - r_0)} \right\} ; r_{ij} \geq r_0 \\ &= \prod e^{-\mu(r_{ij} - r_0)} \prod \left\{ 1 - e^{-\lambda(r_{ij} - r_0)} \right\}, \nu = \mu + \lambda \end{aligned}$$

$$\psi = \psi(R) \prod_{\substack{ij=1 \\ i < j}}^3 f(r_{ij})$$

with $f(r_{ij}) = 1 - e^{-\lambda(r_{ij} - r_0)}$

It is hoped that this form of calculation will clear up the discrepancy in the size of the lightest nuclei as found experimentally and theoretically. If this approach fails, other factors will have to be introduced such as the spin-orbit force, and three-body forces.

Appendix A.

Coordinate Systems

In calculations on the lightest nuclei, it is found convenient to separate out the motion of the centre of mass by a change in the coordinate system.

A system of A particles of equal mass is considered with \underline{r}_i the position vector of the i^{th} particle. It is convenient to separate the particles into two groups containing N and Z particles. This notation is used since the groups will usually be of N neutrons and Z protons respectively.

The following general coordinate system separates out the centre of mass motion.

$$\begin{aligned}
 \underline{R}_A &= \frac{1}{A} \sum_{i=1}^A \underline{r}_i & \underline{R}_{PN} &= \frac{1}{Z} \sum_{i=1}^Z \underline{r}_i - \frac{1}{N} \sum_{i=Z+1}^A \underline{r}_i \\
 \underline{R}_{1p} &= \underline{r}_2 - \underline{r}_1 & \underline{R}_{1n} &= \underline{r}_{Z+2} - \underline{r}_{Z+1} \\
 \underline{R}_{2p} &= \underline{r}_3 - \frac{1}{2}(\underline{r}_1 + \underline{r}_2) & \underline{R}_{2n} &= \underline{r}_{Z+3} - \frac{1}{2}(\underline{r}_{Z+1} + \underline{r}_{Z+2}) \\
 & & & \dots\dots\dots (1) \\
 \underline{R}_{kp} &= \underline{r}_{k+1} - \frac{1}{k} \sum_{i=1}^k \underline{r}_i & \underline{R}_{kn} &= \underline{r}_{Z+k+1} - \frac{1}{k} \sum_{i=Z+1}^{Z+k} \underline{r}_i \\
 \underline{R}_{Z-1,p} &= \underline{r}_Z - \frac{1}{Z-1} \sum_{i=1}^{Z-1} \underline{r}_i & \underline{R}_{N-1,n} &= \underline{r}_{Z+N} - \frac{1}{N-1} \sum_{i=Z+1}^{Z+N-1} \underline{r}_i
 \end{aligned}$$

This transforms the A coordinates $\underline{r}_1 \underline{r}_2 \dots \underline{r}_A$ into the A coordinates

$$\underline{R}_A, \underline{R}_{PN}, \underline{R}_{1p}, \text{----} \underline{R}_{Z-1,p}, \underline{R}_{1n}, \text{-----} \underline{R}_{N-1,n}.$$

It is easily shown that

$$\sum_{i=1}^A \nabla_i^2 = \frac{1}{A} \nabla_{R_A}^2 + \frac{A}{NZ} \nabla_{R_{PN}}^2 + \sum_{k=1}^{Z-1} \left(1 + \frac{1}{k}\right) \nabla_{R_{kp}}^2 + \sum_{k=1}^{N-1} \left(1 + \frac{1}{k}\right) \nabla_{R_{kn}}^2$$

..... (2)

The kinetic energy operator for a system of A nucleons is

$$-\frac{\hbar^2}{2m} \sum_{i=1}^A \nabla_i^2$$

The transformation (1) splits up the kinetic energy operator into the following parts:-

$$-\frac{\hbar^2}{2m} \frac{1}{A} \nabla_{R_A}^2 \quad - \text{ Kinetic energy of centre of mass.}$$

$$-\frac{\hbar^2}{2m} \frac{A}{NZ} \nabla_{R_{PN}}^2 \quad - \text{ Kinetic energy of centre of mass of protons relative to centre of mass of neutrons.}$$

$$-\frac{\hbar^2}{2m} \sum_k \left(1 + \frac{1}{k}\right) \nabla_{R_{kp}}^2 \quad - \text{ Kinetic energy of relative motion of protons.}$$

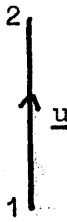
$$-\frac{\hbar^2}{2m} \sum_k \left(1 + \frac{1}{k}\right) \nabla_{R_{kn}}^2 \quad - \text{ Kinetic energy of relative motion of neutrons.}$$

The various coordinate systems used in considering the two three and four body systems, are listed below. They are all special cases of (1).

In addition, the expression is given for

$$\rho_A^2 = \sum_{\substack{i,j=1 \\ i < j}}^A \underline{r}_{ij}^2 \quad \dots\dots\dots (3)$$

Two Body



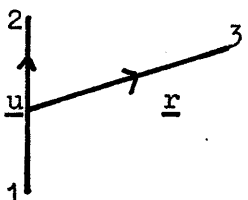
$$\underline{z} = N = 1$$

$$\underline{u} = \underline{r}_2 - \underline{r}_1 \quad ; \quad \underline{R}_2 = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$$

$$\sum_{i=1}^2 \nabla_i^2 = 2 \nabla_u^2 + \frac{1}{2} \nabla_{R_2}^2$$

$$\rho_2^2 = u^2 \quad \dots\dots\dots (4)$$

Three Body



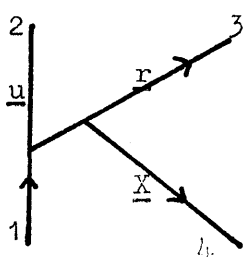
$$\underline{z} = 2 \quad N = 1$$

$$\underline{u} = \underline{r}_2 - \underline{r}_1 \quad ; \quad \underline{r} = \underline{r}_3 - \frac{1}{2}(\underline{r}_1 + \underline{r}_2); \underline{R}_3 = \frac{1}{3}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3)$$

$$\sum_{i=1}^3 \nabla_i^2 = 2 \nabla_u^2 + \frac{3}{2} \nabla_r^2 + \frac{1}{3} \nabla_{R_3}^2$$

$$\rho_3^2 = \frac{3}{2} u^2 + 2 r^2 \quad \dots\dots\dots (5)$$

Four Body (1) $\underline{z} = 3 \quad N = 1$



$$\underline{u} = \underline{r}_2 - \underline{r}_1 \quad ; \quad \underline{r} = \underline{r}_3 - \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$$

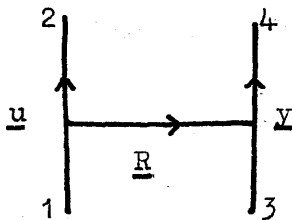
$$\underline{x} = \underline{r}_4 - \frac{1}{3}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3); \underline{R}_4 = \frac{1}{4}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4)$$

$$\sum_{i=1}^4 \dot{v}_i^2 = 2 \dot{v}_u^2 + \frac{3}{2} \dot{v}_r^2 + \frac{4}{3} \dot{v}_x^2 + \frac{1}{4} \dot{v}_{R_4}^2$$

$$\rho_{41}^2 = 2u^2 + \frac{8}{3} r^2 + 3x^2$$

..... (6)

Four Body (2) $N = Z = 2$



$$\underline{u} = \underline{r}_2 - \underline{r}_1 ; \quad \underline{u} = \underline{r}_4 - \underline{r}_3$$

$$\underline{R} = \frac{1}{2}(\underline{r}_3 + \underline{r}_4) - \frac{1}{2}(\underline{r}_1 + \underline{r}_2) ; \quad R_4 = \frac{1}{4}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4)$$

$$\sum_{i=1}^4 \dot{v}_i^2 = 2 \dot{v}_u^2 + 2 \dot{v}_y^2 + \dot{v}_R^2 + \frac{1}{4} \dot{v}_{R_4}^2$$

$$\rho_{42}^2 = 2u^2 + 2y^2 + 4R^2$$

..... (7)

Appendix B.Generalised Sum Rule.

The electric 2^1 -pole operator D_1 does not depend explicitly on the nuclear interaction (Siegert 1937, Sachs and Austern 1951).

The generalised oscillator strength f_{ab}^1 is defined by

$$f_{ab}^1 = \frac{2m}{\hbar^2} (E_b - E_a) |(D_1)_{ab}|^2$$

The Schrodinger equations for ψ_a and ψ_b^* are

$$H \psi_a = E_a \psi_a \quad ; \quad H \psi_b^* = E_b \psi_b^*$$

Hence

$$\begin{aligned} (E_b - E_a) \int \psi_b^* D_1 \psi_a d\tau &= \int \psi_b^* \{ H D_1 - D_1 H \} \psi_a d\tau \\ &= \int \psi_b^* [H, D_1] \psi_a d\tau \end{aligned}$$

$$\text{i.e. } (E_b - E_a)(D_1)_{ab} = [H, D_1]_{ab}$$

Now

$$\begin{aligned} f_{ab}^1 &= \frac{2m}{\hbar^2} (E_b - E_a) (D_1)_{ba} (D_1)_{ab} \\ &= \frac{m}{\hbar^2} \left\{ (D_1)_{ba} (E_b - E_a) (D_1)_{ab} - (E_a - E_b) (D_1)_{ba} (D_1)_{ab} \right\} \end{aligned}$$

Hence summing over all final states b and applying the closure relation

(II.6) gives the generalised sum rule

$$\sum_b f_{ab}^1 = -\frac{m}{\hbar^2} \left[[H, D_1], D_1 \right]_{aa}$$

$$= -\frac{m}{\hbar^2} \langle \psi_a | \left[[H, D_1], D_1 \right] | \psi_a \rangle$$

Since $\langle \psi_a | \psi_a \rangle = 1$ and $\langle \psi_a | H | \psi_a \rangle = E_a$, we have

$$\sum_b f_{ab}^1 = -\frac{m}{\hbar^2} \left(\langle \psi_a | H D_1^2 | \psi_a \rangle - 2 E_a \langle \psi_a | D_1^2 | \psi_a \rangle \right)$$

$$= \frac{m}{\hbar^2} \left(\langle \psi_a | D_1^2 H | \psi_a \rangle - 2 E_a \langle \psi_a | D_1^2 | \psi_a \rangle \right)$$

$$= \frac{m}{\hbar^2} \left(\langle \psi_a | D_1^2 H | \psi_a \rangle - 2 E_a \langle \psi_a | D_1^2 | \psi_a \rangle \right)$$

Now

..... (

where f_{ab}^1 is the particular case of (2) where

$$a=0, b=1, c=1, d=0, \gamma=1$$

$$f_{ab}^1 = \frac{m}{\hbar^2} \left(\langle \psi_a | D_1^2 H | \psi_a \rangle - 2 E_a \langle \psi_a | D_1^2 | \psi_a \rangle \right)$$

Hence on simplification

$$f_{ab}^1 = \frac{m}{\hbar^2} \left(\langle \psi_a | D_1^2 H | \psi_a \rangle - 2 E_a \langle \psi_a | D_1^2 | \psi_a \rangle \right)$$

$$[H, D_1^2]$$

$$= \langle \psi_a | [H, D_1^2] | \psi_a \rangle$$

Appendix C. Evaluation of I (a; p q r)

In Chapter II§9 (69) the integrals

$$I(a; p q r) = \int_0^1 \frac{t^p (1-t^2)^q}{(\alpha+t)^r} dt \dots\dots\dots (1)$$

were introduced. This integral is a particular case of the integral given by Grobner and Hofreiter (1950) Vol. II p. 175.

$$\begin{aligned} & \int_a^b \frac{(x-a)^{K-1} (b-x)^{\lambda-1} (x+\gamma)^m}{(cx+d)^{K+\lambda+n}} dx \\ &= \frac{(b-a)^{K+\lambda-1}}{(bc+d)^K (ac+d)^{\lambda}} \sum_{v=0}^n \frac{B(K+v, \lambda+n-v)}{(bc+d)^v (ac+d)^{n-v}} \sum_{\mu=0}^v m C_{\mu}^{n-m} C_{v-\mu}^{\lambda} (b+\gamma)^{\mu} (a+\gamma)^{m-\mu} \\ & \quad (ac+d) \cdot (bc+d) > 0 \quad \lambda > 0 \quad m \leq n \quad m, n = 0, 1, 2, \dots \\ & \quad K > 0 \quad \dots\dots\dots (2) \end{aligned}$$

Then I (a; p q r) is the particular case of (2) where

$$a = 0 \quad b = 1 \quad c = 1 \quad d = \alpha \quad \gamma = 1$$

$$K = p+1 \quad \lambda = q+1 \quad m = q \quad n = r-p-q-2$$

Hence on simplification

$$\begin{aligned} & I(a; p q r) \\ &= \frac{1}{(\alpha+1)^{p+1}} \frac{1}{\alpha^{q+1}} \sum_{\mu=0}^q \sum_{v=\mu}^{q+\mu} \frac{B(p+v+1, \lambda+n-v)}{(\alpha+1)^v \alpha^{n-v}} C_{\mu}^q C_{v-\mu}^{n-q} 2^{\mu} \\ & \quad n = r-p-q-2 \quad \dots\dots\dots (3) \end{aligned}$$

with $\alpha > 0 \quad p+1 > 0 \quad q+1 > 0 \quad r \geq p+2q+2$

The integral $I(a; p q r)$ is given in Table C1 for various values of $p q r$ in the form

$$I(a; p q r) = \frac{1}{A} \frac{1}{a} \frac{1}{(a+1)^m} f(a) \dots\dots\dots (4)$$

Table C1

p	q	r	l	m	A	$f(a)$
1	2	7	5	4	6.5	$5a^2 + 4a + 1$
1	2	8	6	5	7.6.5	$35a^3 + 47a^2 + 25a + 5$
1	4	12	10	7	11.10.7.3	$231a^5 + 593a^4 + 686a^3 + 434a^2 + 147a + 21$
2	2	8	5	5	7.5.3	$8a^2 + 5a + 1$
2	2	9	6	6	7.6.5.4.	$64a^3 + 69a^2 + 30a + 5$
2	2	10	7	7	9.7.5.4	$96a^4 + 147a^3 + 101a^2 + 35a + 5$
3	2	9	5	6	7.6.5.4.	$35a^2 + 18a + 3$
3	2	10	6	7	7.6.4.3	$21a^3 + 19a^2 + 7a + 1$
3	2	11	7	8	9.8.7.5	$105a^4 + 136a^3 + 80a^2 + 24a + 3$
3	2	12	8	9	11.9.8.7.5	$1155a^5 + 1947a^4 + 1584a^3 + 736a^2 + 189a + 21$
3	3	12	8	8	11.10.7.4.3	$231a^4 + 312a^3 + 186a^2 + 56a + 7$
3	3	13	9	9	11.10.9.7.4	$693a^5 + 1245a^4 + 1041a^3 + 489a^2 + 126a + 14$
3	3	14	10	10	13.12.11.10.	$429a^6 + 962a^5 + 1040a^4 + 670a^3 + 265a^2 + 60a + 6$
3	4	14	10	9	13.11.10.9.2	$429a^5 + 789a^4 + 666a^3 + 314a^2 + 81a + 9$
5	2	12	6	9	11.9.7.5.4	$231a^3 + 159a^2 + 45a + 5$
5	2	13	7	10	11.10.9.7.4	$462a^4 + 460a^3 + 211a^2 + 50a + 5$
5	2	14	8	11	13.11.10.9.4	$858a^5 + 1118a^4 + 715a^3 + 265a^2 + 55a + 5$
5	3	14	8	10	13.11.9.8.5	$429a^4 + 450a^3 + 210a^2 + 50a + 5$
7	2	14	6	11	13.11.9.8.5	$429a^3 + 239a^2 + 55a + 5$

Appendix D. Alpha Particle Integrals.

The integrals defined in V§3 have been evaluated for the four-body case and are given as follows.

Using the definitions

$$\mu_{ij} = \mu_i + \mu_j \quad ; \quad \eta_{ij} = 2\mu_{ij} + \lambda$$

then the kinetic energy integrals are:-

$$I_1(\mu_i | \mu_j) = -\frac{9}{16} \pi^{9/2} \frac{\mu_i \mu_j}{\mu_{ij}^{11/2}}$$

$$I_2(\mu_i | \mu_j) = -\frac{\pi^3}{\mu_{ij}^{23/2}} \cdot \frac{5.67}{2^{10}} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{3}{2}\right) \mu_j \left\{ 3\mu_i - 4\mu_j \right\}$$

$$I_4(\mu_i | \mu_j) = I_4^{(2)}(\mu_i | \mu_j) \mu_j^2 + I_4^{(1)}(\mu_i | \mu_j) \mu_j + I_4^{(0)}(\mu_i | \mu_j)$$

where

$$I_4^{(2)}(\mu_i | \mu_j) = -8 \frac{1}{\mu_{ij}^2} J_3(\mu_i | \mu_j | 0)$$

$$I_4^{(1)}(\mu_i | \mu_j) = -132 J_3(\mu_i | \mu_j | 0)$$

$$I_4^{(0)}(\mu_i | \mu_j) = \frac{\pi^3}{\mu_{ij}^{31/2}} \Gamma\left(\frac{11}{2}\right) \Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{3}{2}\right) \frac{693117727}{3 \cdot 2^{28}}$$

The potential energy integrals are

$$J_1(\mu_i | \mu_j | \lambda) = \frac{\pi^{9/2}}{2^{9/2} \mu_{ij}^3 \eta_{ij}^{3/2}}$$

$$J_2 (\mu_i | \mu_j | \lambda) = \frac{\pi^3}{\mu_{ij}^{4\pi} 5/2} \frac{\Gamma(\frac{7}{2}) \Gamma(\frac{5}{2}) \Gamma(\frac{3}{2})}{2^{27/2}} \\ \times \left\{ \frac{693}{\eta_{ij}^4} + \frac{462}{\eta_{ij}^3} \mu_{ij} + \frac{1519}{2 \eta_{ij}^2} \mu_{ij}^2 + \frac{2005}{6 \eta_{ij}} \mu_{ij}^3 + \frac{6965}{16} \mu_{ij}^4 \right\}$$

$$J_3 (\mu_i | \mu_j | \lambda) = \frac{\pi^3}{\eta_{ij}^{7/2} \mu_{ij}^{13}} \frac{\Gamma(\frac{3}{2})}{2^{39/2}} \sum_{n=0}^8 a_n \left(\frac{\mu_{ij}}{\eta_{ij}} \right)^n \Gamma\left(n + \frac{7}{2}\right)$$

with

$$a_0 = \frac{51730077}{2^{13}} \Gamma\left(\frac{13}{2}\right) ; \quad a_1 = \frac{528457}{2^8} \Gamma\left(\frac{13}{2}\right)$$

$$a_2 = \frac{22711 \cdot 697}{5 \cdot 3 \cdot 2^8} \Gamma\left(\frac{11}{2}\right) ; \quad a_3 = \frac{592817}{5 \cdot 3 \cdot 2^5} \Gamma\left(\frac{11}{2}\right)$$

$$a_4 = \frac{3585167}{7 \cdot 5 \cdot 2^6} \Gamma\left(\frac{9}{2}\right) ; \quad a_5 = \frac{29647}{7 \cdot 5 \cdot 2^2} \Gamma\left(\frac{9}{2}\right)$$

$$a_6 = \frac{2757}{5 \cdot 2^2} \Gamma\left(\frac{7}{2}\right) ; \quad a_7 = \frac{26}{3} \Gamma\left(\frac{7}{2}\right) ; \quad a_8 = \Gamma\left(\frac{7}{2}\right)$$

The normalisation integrals are

$$J_1 (\mu_i | \mu_j | 0) = \frac{\pi}{2^6 \mu_{ij}^{9/2}} \frac{9/2}{\mu_{ij}^{9/2}}$$

$$J_2 (\mu_i | \mu_j | 0) = \frac{\pi^3}{\mu_{ij}^{21/2}} \frac{5 \cdot 67}{3} \frac{\Gamma(\frac{7}{2}) \Gamma(\frac{5}{2}) \Gamma(\frac{3}{2})}{2^{13}}$$

$$J_3 (\mu_i | \mu_j | 0) = \frac{\pi^3}{\mu_{ij}^{33/2}} \frac{\Gamma(\frac{11}{2}) \Gamma(\frac{7}{2}) \Gamma(\frac{3}{2})}{5 \cdot 2^{22}} \cdot 293127$$

The r.m.s. radius integrals can be obtained using

$$R_n (\mu_i | \mu_j) = - \frac{d}{d\mu_{ij}} J_n (\mu_i | \mu_j | 0) \quad n = 1, 2, 3$$

Finally the Coulomb energy integrals are

$$C_1 (\mu_i | \mu_j) = \frac{\pi^4}{2^{9/2} \mu_{ij}^4}$$

$$C_2 (\mu_i | \mu_j) = \frac{\pi^3}{\mu_{ij}^{10}} \frac{\Gamma(\frac{5}{2}) \Gamma(\frac{3}{2})}{3 \cdot 2^{41/2}} \cdot 182011$$

$$C_3 (\mu_i | \mu_j) = \frac{\pi^3}{\mu_{ij}^{16}} \frac{\Gamma(\frac{7}{2}) \Gamma(\frac{3}{2})}{5 \cdot 2^{75/2}} \cdot 524016460629$$

References: (1) J. J. Sakurai, *Modern Quantum Mechanics*, McGraw-Hill, New York, 1967, p. 110.

(2) R. H. Dicke, *Phys. Rev.*, **92**, 1108 (1953).

(3) J. J. Sakurai, *Modern Quantum Mechanics*, McGraw-Hill, New York, 1967, p. 110.

Baker, R. G., Robertson, R. L. (1957) *Proc. Phys. Soc.*, **70**, 110.

Olson, A. G. (1954) *Proc. Phys. Soc.*, **67**, 20.

Chandrasekhar, S. (1955) *Phil. Mag.*, **1**, 10.

Bibliography

- Abraham G, Cohen L, Roberts A.S., (1955) Proc. Phys. Soc. A68, 265.
- Austern N. (1952) Phys. Rev. 85, 283.
- Austern N. (1953) Phys. Rev. 92, 670.
- Bellamy E.H., McWhirter R.W.P., Palit P, (1957) Proc. Phys. Soc. A 70, 763.
- Bellamy E.H., Palit P, (1958) Proc. Phys. Soc. 72, 880.
- Benedict T.S., Woodward W.M., (1951) Phys. Rev. 83, 1269.
- Bethe H.A., Longmire C. (1950) Phys. Rev. 77, 647.
- Biedenharn L.C., Blatt J.M., Kalos M.H. (1958) Nuc. Phys. 6, 359.
- Blatt J.M., Derrick G. (1958) Nuc. Phys. 8, 602.
- Blatt J.M., Weisskopf V.F. (1952) Theoretical Nuclear Physics (New York; John Wiley) Chapter XII, Appendix B.
- Bransden B.H., Robertson H.H. (1958) Proc. Phys. Soc. 72, 770.
- Bransden B.H., Robertson H.H., Swan P. (1956) Proc. Phys. Soc. A 69, 877.
- Bronnan J.G., Sachs R.G. (1952) Phys. Rev. 88, 824.
- Brown F.W. (1939) Phys. Rev. 56, 1107.
- Burke P.G., Robertson H.H. (1957) Proc. Phys. Soc. A 70, 777.
- Clark A.C. (1954) Proc. Phys. Soc. A 67, 323.
- Cranberg L. (1958) Bull. Am. Phys. Soc. Ser. II, 3, 173.
- Dabrowski J. (1958) Proc. Phys. Soc. 71, 658.
- de Saussure G., Osborne L.S. (1955) Phys. Rev. 99, 843.
- de Swart J. J., Marshak R.E. (1958) Phys. Rev. 111, 272.
- Emery V.J. (1958) Nuc. Phys. 6, 585.
- Ferguson G.A., Halpern J., Nathans R., Yergin P.F. (1954) Phys. Rev. 95, 776.

- Feshbach H., Rubinow I. (1955) Phys. Rev. 98, 188.
- Feshbach H., Schwinger J. (1951) Phys. Rev. 84, 194.
- Flowers B.H., Mandl F. (1951) Proc. Roy. Soc. A 206, 131.
- Foldy L.L. (1957) Phys. Rev. 107, 1303.
- Fowler W.A., Lauritsen C.C., Tollestrup A.V. (1949) Phys. Rev. 76, 1767.
- Fox L., Goodwin E.T. (1949) Proc. Camb. Phil. Soc. 45, 373.
- Frankel S. (1955) Phys. Rev. 99, 169.
- Frohlich H., Huang K., Sneddon I.N. (1947) Proc. Roy. Soc. A 191, 61.
- Fuller E.G. (1954) Phys. Rev. 96, 1306.
- Gaerttner E.R., Yeater M.L. (1951) Phys. Rev. 83, 146.
- Gamba A. (1951) Nuovo Cim. 8, 605.
- Gamba A., Malvano R., Radicati L.A. (1952) Phys. Rev. 87, 440.
- Gammel J.L., Thaler R.M., (1957) Phys. Rev. 107, 291, 1337.
- Gartenhaus S. (1955) Phys. Rev. 100, 900.
- Gerjuoy E., Schwinger J. (1942) Phys. Rev. 61, 138.
- Goldemberg J., Lopes J.L. (1954) Nuovo Cim. 12, 817.
- Goldhaber M., Teller E. (1948) Phys. Rev. 74, 1046.
- Gorbunov A.N., Spiridonov V.M. (1958) Soviet Physics JETP 6, 16.
- Griffiths G.M., Warren J.B. (1955) Proc. Phys. Soc. A 68, 781.
- Grobner, W., Hofreiter N. (1950) Integraltafel II Bestimmte Integrale
(Vienna; Springer-Verlag)
- Gunn J.C., Irving J. (1951) Phil. Mag. 42, 1353.
- Hofstadter R. (1956) Rev. Mod. Phys. 28, 214.
- Hu T-M., Hsu K-N (1951) Proc. Roy. Soc. A 204, 476.
- Hulthén L., Sugawara N. (1957) Handbuch der Physik. Vol. XXXIX.
ed. S. Flügge (Berlin; Springer-Verlag).

- Irving J. (1953) Proc. Phys. Soc. A 66, 17.
- Iwamoto F., Yamada M. (1957) Prog. Theor. Phys. 17, 543; 18, 345.
- Jastrow R. (1955) Phys. Rev. 98, 1479.
- Kikuchi S. (1952) Phys. Rev. 86, 126.
- Kikuta T., Morita M., Yamada M. (1956) Prog. Theor. Phys. 15, 222.
(1957) Prog. Theor. Phys. 17, 326.
- Levinger J.S., Bethe H.A. (1950) Phys. Rev. 78, 115.
- Levinger J.S. (1951) Phys. Rev. 84, 43.
- Levinger J.S. (1956) Phys. Rev. 101, 733.
- Levinger J.S., Austern N., Morrison P. (1957) Nuc. Phys. 3, 456.
- Levinger J.S., Rustgi M.L. (1957) Phys. Rev. 106, 530.
- Levinger J.S. (1958) Nuc. Phys. 8, 428.
- Levy M.M. (1952) Phys. Rev. 88, 725.
- McAllister R.W., Hofstadter R. (1956) Phys. Rev. 102, 851.
- MacDonald W.M. (1955) Phys. Rev. 98, 60.
- Marshall J.F., Guth E. (1950) Phys. Rev. 78, 738.
- Morpurgo G. (1952) Nuovo Cim. 9, 461.
- Moskowsk*i* S.A. (1955) Beta- and Gamma - Ray Spectroscopy ed K. Siegbahn
(Amsterdam; North Holland) Chapter XIII.
- Pease R.L., Feshbach H. (1951) Phys. Rev. 81, 142.
(1952) Phys. Rev. 88, 945.
- Perry J.E., Bame S.J. (1955) Phys. Rev. 99, 1368.
- Preston M.A., Bird J.R. (1955) Can. Jour. Phys. 33, 399.

- Preston M.A., Shapiro J. (1956) Can. Jour. Phys. 34, 451.
 (1957) Can. Jour. Phys. 35, 451.
- Preston M.A., Preston R.L. (1958) Can. Jour Phys. 36, 579.
- Radicati L.A. (1953) Proc. Phys. Soc. A 66, 139.
- Rand S. (1957) ~~Phys.~~ Rev. 107, 208.
- Reid J.M., Swinbank P., Atkinson J.R. (1956) Physica 22, 1142.
- Sachs R.G., Austern N. (1951) Phys. Rev. 81, 705.
- Schiff L.I. (1949) Quantum Mechanics (New York ; McGraw - Hill)
- Schiff L.I. (1950) Phys. Rev. 78, 733.
- Siegert A.J.F. (1937) Phys. Rev. 52, 787.
- Signell P.S., Marshak R.E. (1957) Phys. Rev. 106, 832.
 (1958) Phys. Rev. 109, 1229.
- Verde M. (1950) Helv. Phys. Acta. 23, 453.
- Wheeler J.A. (1937) Phys. Rev. 52, 1107.
- Wilkinson D.H. (1953) e.g. Phil. Mag. 44, 542, 1019.
 (1956) Physica 22, 1039.
- Yoshida S. (1951) Prog. Theor. Phys. 6, 1032.
- Zachariasen F. (1956) Phys. Rev. 101, 371.